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Victor Anandam

Harmonic Functions and Potentials on Finite or Infinite Networks

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Preface

The aim of the current work is to present an autonomous theory of harmonic functions on infinite networks akin to potential theory on locally compact spaces as developed primarily by Brelot (without sanctioning any explicit role for the derivatives of functions defined on the space). Though random walks and electrical networks are two important sources for the advancement of the present theory, neither probabilistic methods nor energy integral techniques are used here to prove the results in an infinite network. The relevance of this study is partly because in many infinite networks (like homogeneous trees, for example), any real-valued function defined on the network is a difference of two superharmonic functions.

We consider principally the classification theory of infinite networks based on the existence of Green functions, bounded harmonic functions etc., and then balayage, equilibrium principle, domination principle, Schrödinger operators, polyharmonic functions and the Riesz-Martin compactification of the network. An important feature is the study of parabolic networks. These are the networks on which no positive potentials exist or equivalently, these are the networks on which the Green function cannot be defined. On parabolic networks we investigate the properties of pseudo-potentials (analogous to logarithmic potentials on the complex plane) introduced via a development of a notion of flux.

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Ramanujan Institute for Advanced Study in Mathematics *Victor Anandam*
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March 2011

Contents

1 Laplace Operators on Networks and Trees 1
1.1 Introduction 1
1.2 Preliminaries 4
1.2.1 Examples of Superharmonic Functions on Networks 8
1.3 Green's Formulas 10
1.4 Minimum Principle 13
1.5 Infinite Trees 16

2 Potential Theory on Finite Networks 21
2.1 Incidence Matrix, Kirchhoff's Problem 21
2.2 Dirichlet-Poisson Equations in Finite Networks 23
2.3 Dirichlet Semi-Norm 32
2.4 Schrödinger Operators on Finite Networks 34

3 Harmonic Function Theory on Infinite Networks 45
3.1 Infinite Networks and the Laplace Operator 46
3.2 Classification of Infinite Networks 52
3.2.1 Harmonic Measure at Infinity of a Section 54
3.2.2 Positive Harmonic Functions on a Network 58
3.2.3 Integral Representation of Positive Harmonic Functions 60
3.3 Hyperbolic Networks 62
3.4 Parabolic Networks 70
3.5 Flux at Infinity 78
3.6 Pseudo-Potentials 87

4 Schrödinger Operators and Subordinate Structures on Infinite Networks 91
4.1 Local Properties of q-Superharmonic Functions 92
4.2 Classification of q-Harmonic Networks 101
4.3 Subordinate Structures 103

5 Polyharmonic Functions on Trees 109
5.1 Polyharmonic Functions on Infinite Trees 111
5.2 Polyharmonic Functions with Point Singularity 121
5.3 Riesz-Martin Representation for Positive m-Superharmonic Functions 129

References 133

Index 139

Chapter 1
Laplace Operators on Networks and Trees

Abstract This work is an autonomous study of functions on infinite networks reflecting potential theory on locally compact spaces, influenced by the function theory associated with random walks and electrical networks. Starting with an overview of the contents of the five chapters presented here, this chapter introduces harmonic and superharmonic functions and their basic properties in networks. A discrete version of the Green's formula is given and the Minimum Principle for superharmonic functions is proved. Infinite trees as a special case are seen to provide examples and motivations for the development of an abstract discrete function theory on infinite networks.

1.1 Introduction

A *graph* consists of a finite number of points (called vertices) and a finite number of lines (called edges) joining some of them. The graph theory studies the inter-relation between the vertices and the edges (for example, [66]). Now for some problems, the edges have to be oriented in which case the graph is called a *digraph*. It would be easier to represent a digraph by its *incidence matrix* of order $n \times m$, where m is the number of edges and n is the number of vertices, with entries $-1, 0$ or 1. The interest in graph theory comes from the fact that many real-life situations can be represented as graphs.

Take for example, the postman problem: The postman collects the post from the post office and walks through all the streets in his beat, distributing the letters and finally returns to the post office. His problem is how well to choose a route so that, if possible, he does not go through any street more than once, yet covers all the streets. To solve this, we can think of each street corner as a vertex and each street as an edge, thus getting the model of a graph; the problem now reduces to finding a path that contains all the edges once and once only. Like this, there are other problems connected with chemical bonding, bus routes, work assignments etc. In some situations like bus routes, the distance between two vertices (that is, between

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two bus stops) may be important. That is, each edge has a real number associated with it and then we have *weighted graphs*. It is interesting to study these *geometrical structures* of a graph for their own merit. But it would be more fruitful to represent a *physical problem* as a graph theory problem and try to solve it.

Though graph theory generally deals with a finite number of objects and their inter-connectedness where the geometrical aspects of graphs play a decisive role, yet there are also problems that involve functions on finite graphs. For example, consider a *finite electrical network*. This can be represented as a graph [32] provided with a voltage-current regime subject to *Ohm's law* and *Kirchhoff's voltage and current laws*. Here we are interested not only in the geometrical properties of a finite graph but also on functions defined on nodes and branches satisfying certain conditions. In this context, the incidence matrix of the graph takes care of the geometrical properties of the graph and for the function-theoretic aspects one introduces the *Laplace operator* Δ dependent on the incidence matrix and its transpose which can be considered as operators on functions defined on its edges and vertices.

There is another development which requires the study of *infinite graphs*. Consider finite difference approximations of equations in physics; some of them lead to partial difference equations [14]. The approximations to find a solution involve horizontal and vertical displacements and so can be treated as functions on an infinite grid in the context of electrical networks. Take for example, *wave equations*; the domain of existence of the solution may be unbounded, suggesting a problem in a graph with infinite vertices [73]. Another example of an infinite graph arises in the study of *Markov chains* [68]. A Markov chain consists of a countable number of states provided with a *transition probability* and the *Markov property* which says that given the present, the past and the future are independent.

The study of functions on infinite networks has thus far been carried out on the background of Markov chains and random walks or on the requirements of extending results from finite electrical networks to infinite networks. There are many common features in these two developments. Actually, a close connection has been established between the concepts like transition probabilities, transience, recurrence, hitting time etc. used in the probabilistic study of functions on infinite networks and the concepts like effective resistance, equilibrium principle, capacity etc. used in a current-voltage regime in electrical networks. The effective resistance has a close relation to the escape probability for a reversible Markov chain [59, 64] which is characterized by the transition probability from one state to another. The similarity between the conductance and the transition probability is obvious. Consequently, it is not uncommon to see a problem arising in the context of random walks being solved by electrical methods and conversely. The electrical methods make use of functional analysis techniques, starting with the Dirichlet norm (the discrete analogue of the energy integral in the classical case) and its associated inner-product. Thus, the abstract potential theory on infinite networks, as developed by Yamasaki [70], Soardi [63] and others, is a study of Dirichlet finite functions (modeled after Dirichlet functions in the classical potential theory) dealing with discrete analogues of the solution of Poisson equation, Green's function, extremal length, Royden decomposition and Royden compactification.

The current work presents an autonomous study of functions on infinite networks influenced by potential theory on locally compact spaces which does not assign any direct role for the notion of derivatives of functions. Initially, finite networks provided with the Laplacian operator are taken up for the study, the development depending on algebraic methods starting with the incidence matrix. We can call this *algebraic potential theory* because of the association of the Laplacian (represented as a matrix) with electric potentials [24]. Later, infinite networks, with the Laplace operator defined as in the finite case, are taken up for consideration. In this situation, the development resembles the study of harmonic and subharmonic functions in the complex plane or more generally in $\Re^n, n \geq 2$ [10, 27, 53], and in the Brelot axiomatic potential theory [17, 28, 40]. Here the Dirichlet norm does not play a dominant role; nor are the probabilistic interpretations considered. However, in both the finite and the infinite cases, the important basic properties and significant results like the equilibrium principle, the condenser principle, the capacity etc. that are related to an electrical network come as solutions to the following *Dirichlet-Poisson problem* on the (finite or infinite) graph X, namely: Let F be a subset of the vertex set X. Suppose f and g are real-valued functions on X. Then, find u defined on the vertex set X satisfying the conditions $\Delta u = f$ at each vertex in F and $u = g$ on $X \setminus F$.

The present work is rather like a discrete version of function theory on Riemann surfaces [3, 39, 58, 65] and Riemannian manifolds [60], devoid of any attempt to connect it to any of the many important works on electrical networks and random walks. We develop a function theory on networks similar to the classical and the axiomatic potential theory on Euclidean spaces and on locally compact spaces. The basic definitions of potentials, Green's kernel, balayage etc. are introduced here as in the case of the Brelot axiomatic theory rather than as in the theory of probability ([31] for example). Yamasaki [69, 70] also has proved many potential theoretic results in an infinite network without involving the methods used in the study of random walks. However his study is based on Dirichlet norms and functional analysis methods, resembling potential theory on Dirichlet spaces studied by Deny [41, 42], Beurling and Deny [22, 23], Fukushima [46], Bouleau and Hirsch [26] and others. These methods are not convenient if we have to study potential theory on infinite networks in which the only non-negative potential is 0. Thus a deeper study of infinite networks without positive potentials as in the case of parabolic Riemann surfaces becomes cumbersome. Under these circumstances, the approach we have adopted here is easy to deal with situations in infinite networks that resemble those of Riemann surfaces that are hyperbolic or parabolic.

Chapter 1 is devoted to some preliminary remarks concerning networks and trees, the interior and the boundary of subsets in a network, inner and outer normal derivatives, Green's formulas, the definition and some properties of superharmonic functions and the minimum principle.

Chapter 2 brings into focus certain aspects of potential theory on *finite networks*. The Laplacian is represented by a matrix and the properties of this matrix lead to the minimum principle, domination principle, equilibrium principle and solutions to some mixed boundary-value problems like Poisson-Dirichlet problem and Neumann problem in a finite network. It is easy then to consider in a similar vein the Schrödinger operators in finite networks. These results in a finite network are already proved in Bendito et al. [20] by assuming the symmetry of conductance and then constructing equilibrium measures appropriate to each principle. Ours is a unified method based on the inverses of certain sub-matrices of the Laplace matrix.

Chapter 3 deals with the classification theory of infinite networks. It starts with the first broad division of networks into *hyperbolic and parabolic networks*, depending on whether it is possible or not to define the *Green kernel* on the network. A hyperbolic network is further classified based on the existence of non-constant positive and bounded harmonic functions on the network. This leads to the *Riesz-Martin representation* of positive superharmonic functions on a hyperbolic network. Later a study of parabolic networks is taken up, starting with the construction of a kernel like $\log |x - y|$ in the plane. The notion of *flux at infinity* of a superharmonic function is discussed in detail. Balayage and Dirichlet problem on arbitrary subsets of a parabolic network receive attention. Then, an introductory study of *pseudo-potentials* (similar to logarithmic potentials in the plane) follows.

Chapter 4 is devoted to the potential theory on an infinite network X associated to the *Schrödinger operator* $\Delta u(x) - q(x)u(x)$, for an arbitrary real-valued function $q(x)$ and then more specifically when $q(x) \geq \dfrac{\Delta\xi(x)}{\xi(x)}$ for some function $\xi > 0$ on X. This condition implies that q can take negative values, but ensures that there exists a positive q-superharmonic function on X. With respect to this operator, the topics like generalised Dirchlet problem, balayage, condenser principle, equilibrium principle, etc. are investigated. This example of two related harmonic structures, one from the Laplace operator and the other from the Schrödinger operator, in the same infinite network is later generalised to study the relation between a basic harmonic structure and a subordinate harmonic structure on X.

Chapter 5 takes up the study of polyharmonic functions on an infinite tree T. A real-valued function s on X is said to be polysuperharmonic of order m or simply m-superharmonic (for an integer $m \geq 1$) if $(-\Delta)^m s \geq 0$. Actually, to characterize m-superharmonic and m-harmonic functions, we build up on the solutions u to the Poisson equation $\Delta u = f$ for an arbitrary real-valued function f on X. Since the solutions to this equation are not easily available in an arbitrary network, we have to confine our study of polyharmonic functions defined on a tree T only. For polysuperharmonic functions on T, Laurent decomposition, m-harmonic Green functions, domination principle, balayage etc. are obtained. Finally, the Riesz-Martin representation for positive m-superharmonic functions is exhibited.

1.2 Preliminaries

Let X be a countable (finite or infinite) set of points, called vertices, some of them pair-wise joined by *edges*; we say that the edge $[x, y]$ joins the vertices x and y. Let Y denote the set of edges which are assumed to be countable. Denote $x \sim y$ to mean that there is an edge $[x, y]$ joining x and y, in which case the vertices x and y are said to be *neighbours*. A vertex e is named *terimal* if it has only one neighbour. A *walk* from x to y is a collection of vertices $\{x = x_0, x_1, ..., x_n = y\}$ where $x_i \sim x_{i+1}$ if $0 \leq i \leq n-1$; for this walk, the *length* is n. If the vertices in the walk are distinct, the walk is referred to as a *path*. The shortest length $d(x, y)$ between x and y is called *the distance between x and y*. We also assume that given any two vertices x and y, there exists *an associated non-negative number*, called *conductance*, $t(x, y) \geq 0$ such that t(x, y) > 0 if and only if $x \sim y$. Then $N = \{X, Y, t\}$ is called *a network* if the following conditions are also satisfied:

1. There is no *self-loop* in N, that is no edge of the form $[x, x]$ in Y.
2. Given any vertices x and y in X, there is a path connecting x and y. (That is, X is *connected*.)
3. Every $x \in X$ has only a finite number of neighbours. (That is, X is *locally finite*.)

Instead of writing $N = \{X, Y, t\}$, we simply write X to refer to a network. If $t(x, y) = t(y, x)$ for every pair of vertices x and y, then we say that X is a network with *symmetric conductance*. A network X is called a *tree* if there is no *cycle* in X, that is there is no closed path $\{x_1, x_2, ..., x_n, x_1\}$ with $n \geq 3$. An infinite tree T is said to be *homogeneous* of degree $q+1$, if each vertex in T has $(q+1)$ neighbours. A tree T is said to be a *standard homogenous tree of degree* $q+1$ if every vertex in T has exactly $(q+1)$ neighbours and $t(x, y) = (q+1)^{-1}$ if $x \sim y$. If a tree T is considered in the context of probability, we denote the conductance as $p(x, y)$ instead of $t(x, y)$, so that $\sum_{y \sim x} p(x, y) = 1$ for any $x \in T$. We refer to $p(x, y)$ as the *transition probability* from x to y. It is important to note that in a tree T, if x and y are any two vertices, then there exists a unique path joining x and y.

For any subset E of a network X, we write $\overset{0}{E} = \{x : x$ and all its neighbours are in $E\}$ and $\partial E = E \setminus \overset{0}{E}$. $\overset{0}{E}$ is referred to as the *interior* of E and ∂E is referred to as the *boundary* of E. This definition of boundary differs from the one used by Chung and Yau [35] and Bendito et al. [18]. According to them, $y \notin E$ is a boundary point of E if and only if there exists a vertex x in E such that $x \sim y$ and the collection of these boundary points is the boundary δE of E. However, the definition of the boundary ∂E given here is preferable in the case of infinite networks, since for many boundary-value problems like the Dirichlet problem the boundary function will be defined on E only. So it is convenient to define the boundary ∂E as a subset of E rather than as a subset lying outside E. Note that for a non-empty subset E, we have $E = \overset{0}{E}$ if and only if $E = X$. An arbitrary set E in X is said to be *circled* if every vertex in ∂E has at least one neighbour in $\overset{0}{E}$. That is, E is circled if and only if $E = \overset{0}{E} \cup \delta \overset{\circ}{E}$, if we use the notation of Bendito et al. [20].

Example: Let e be a fixed vertex. For any vertex x, let $|x|$ denote the distance between e and x. Then $B_m = \{x : |x| \leq m\}$ is circled. For an example of a non-circled set, we can consider in a homogeneous tree of degree $q+1$, $q \geq 2$, the set

E consisting of B_m and one more vertex z, $|z| = m+1$, and the edge connecting z to B_m. Write $V(E)$ to denote the union of E and all the neighbours of each vertex of E, that is $V(E) = E \cup \{y : y \sim x \text{ for some } x \in E\}$. In particular, $V(x)$ denotes the set consisting of x and all its neighbours. Remark that if E is connected, $V(E)$ also is connected. Also note that for any set E, $V(E)$ is circled. For, if $F = V(E)$, then $E \subset \overset{0}{F}$. Hence if $z \in \partial F$, then by definition, z has a neighbour in E and hence in $\overset{0}{F}$. Note that $V(E)$ is the same as $\overline{E} = E \cup \delta E$ in the notations of [20].

For $x \in X$, write $E_1 = V(x)$. Then, E_1 is connected and circled. Let $E_2 = V(E_1)$ which is also connected and circled. Inductively, define $E_n = V(E_{n-1})$ for $n \geq 3$. Then $\{E_n\}$ is an increasing sequence of finite, connected and circled sets such that $\overset{\circ}{E}_{n+1} \supset E_n$ and $X = \bigcup\limits_n E_n$, which is referred to as *a regular exhaustion* of X. For any subset A in X, A^c denotes $X \setminus A$, the complement of A in X.

Proposition 1.2.1. *Let A be circled, and* $B = \overset{\circ}{A}{}^c$. *Then* $\partial B = \partial A$ *and* $\overset{0}{B} = A^c$*; also,* B *is circled.*

Proof. Note $B = \overset{\circ}{A}{}^c = A^c \bigcup \partial A$. Let $z \in \partial A$. Then, for some $y \in \overset{0}{A}$, $y \sim z$; thus $z \in \partial A \subset B$, but a neighbour y of z is not in B, hence $z \in \partial B$. Conversely, let $b \in \partial B$. Then, $b \sim a$ for some $a \in X \setminus B = \overset{0}{A}$. Since $a \in \overset{0}{A}$ and $a \sim b$, we should have $b \in A \setminus \overset{0}{A}$, which means $b \in \partial A$. Consequently, $\partial B = \partial A$ and $\overset{0}{B} = A^c$.

To show that B is circled, take $b \in \partial B$. Since $\partial B = \partial A$, b should have a neighbour $a \in A^c = \overset{0}{B}$. Hence B is circled. □

Proposition 1.2.2. *Let* E *be an arbitrary set and* $F = V(\overset{0}{E})$. *Then* $\overset{0}{E} = \overset{0}{F}$ *and* $\partial F \subset \partial E$*; also* F *is the largest circled set contained in* E.

Proof. By definition, $\overset{0}{E} \subset \overset{0}{F}$. Since $F \subset E$, we also have $\overset{0}{F} \subset \overset{0}{E}$. Hence $\overset{0}{E} = \overset{0}{F}$. To see $\partial F \subset \partial E$, first note that $\partial F \cap \overset{0}{E} = \partial F \cap \overset{0}{F} = \phi$. Then, $\overset{0}{E} \cup \partial F = \overset{0}{F} \cup \partial F = F \subset E = \overset{0}{E} \cup \partial E$ implies that $\partial F \subset \partial E$. $V(\overset{0}{E})$ is circled by definition and $V(\overset{0}{E}) \subset E$.

Suppose the circled set $A \subset E$. Then $\overset{0}{A} \subset \overset{0}{E}$. Let $x \in \partial A$. Then there exists $z \in \overset{0}{A}$ such that $x \sim z$. Since $z \in \overset{0}{E}$ and $z \sim x$, we find that $x \in F$. Hence $\partial A \subset F$. Then, $A = \overset{0}{A} \cup \partial A \subset \overset{0}{E} \cup F = \overset{0}{F} \cup F = F$. □

A lower semi-continuous function $u > -\infty$ on an open set ω in the Euclidean space $\Re^n$, $n \geq 2$, is said to be *superharmonic* on ω if u is not identically ∞ and if for each $a \in \omega$, there exists a closed ball $\overline{B}(a, R) \subset \omega$ such that $u(a) \geq \int\limits_S u(a + r\xi) d\sigma(\xi)$ if $0 < r \leq R$. Here S is the unit sphere and σ is the normalised surface area measure on S, so that $\sigma(S) = 1$. Suppose u is a locally Lebesgue integrable

function on ω such that $\Delta u \leq 0$ *in the sense of distributions*. Then, there exists a superharmonic function v on ω such that $v = u$ a.e. [27, p.43]. In analogy with this result, superharmonic functions on a network are defined as follows:
We consider now functions defined on subsets of a network X. In this presentation, we assume that *all functions are real-valued.*

1. For $x \in X$, and for a function f defined on $V(x)$, define the *Laplacian*

$$\Delta f(x) = \sum_{x \sim x_i} t(x, x_i)[f(x_i) - f(x)] = -t(x)f(x) + \sum_{x \sim x_i} t(x, x_i) f(x_i),$$

   where $t(x) = \sum_{x \sim x_i} t(x, x_i)$; note $t(x) > 0$ for any $x \in X$.
2. We say that f is *harmonic* (respectively *superharmonic, subharmonic*) at x if $\Delta f(x) = 0$ (respectively $\Delta f(x) \leq 0$, $\Delta f(x) \geq 0$). *A function f defined on an arbitrary set E is said to be harmonic (respectively superharmonic, subharmonic) on E if and only if* $\Delta f(x) = 0$ *(respectively* $\Delta f(x) \leq 0$, $\Delta f(x) \geq 0$*) for every* $x \in \overset{0}{E}$.

Remark 1.2.1. 1. Some authors prefer to define a real-valued function u as a harmonic function on E provided its Laplacian is 0 at every vertex of E. This presupposes that u is defined on $V(E)$. But for the topics we discuss here, such as the Dirichlet problem, the minimum principle, the condenser principle etc. u is either not defined outside E or its value outside E is not of consequence. Hence, it becomes important to distinguish between the interior and the boundary vertices of E.

2. It is possible, in the definition of a network X, to leave out the condition that each vertex has only a finite number neighbours. But then, we should assume that $\sum_{y \sim x} t(x, y) < \infty$ for all x in X; also, instead of defining the Laplacian Δf for any real-valued function on f on X, we should confine ourselves to those real-valued functions u on X for which $\sum_{y \sim x} t(x, y)\,|u(y)| < \infty$ for any x in X. In that case, $\sum_{y \sim x} t(x, y)\,[u(y) - u(x)]$ will make sense, since

$$\sum_{y \sim x} t(x, y)\,|u(y) - u(x)| \leq \sum_{y \sim x} t(x, y)\,|u(y)| + |u(x)| \sum_{y \sim x} t(x, y)$$

   for any x in X.

Proposition 1.2.3. *If u and v are superharmonic on a subset E of X and if α, β are two non-negative constants, then $\alpha u + \beta v$ and* $\inf(u, v)$ *are superharmonic on E. Moreover, if u_n is a convergent sequence of superharmonic functions on E such that $u(x) = \lim u_n(x)$ is finite for every vertex in E, then u is superharmonic on E.*

Proof. If $x \in \overset{0}{E}$, then by hypothesis $\Delta u(x) \leq 0$ and $\Delta v(x) \leq 0$. Hence

$$\Delta(\alpha u + \beta v)(x) = \alpha \Delta u(x) + \beta \Delta v(x) \leq 0.$$

Hence, $\alpha u + \beta v$ is superharmonic on X.

To show that $s = \inf(u, v)$ also is superharmonic on E, that is $s(x) = \inf(u(x), v(x))$ for $x \in E$, assume without loss of generality that $s(x) = u(x)$. Then,

$$\begin{aligned}\Delta s(x) &= \sum_{x_i \sim x} t(x, x_i)[s(x_i) - s(x)] \\ &= \sum_{x_i \sim x} t(x, x_i)[s(x_i) - u(x)] \\ &\leq \sum_{x_i \sim x} t(x, x_i)[u(x_i) - u(x)] \\ &= \Delta u(x) \leq 0.\end{aligned}$$

Hence, $s = \inf(u, v)$ is superharmonic on E.

Finally concerning the limit of superharmonic functions, for any $x \in \overset{0}{E}$,

$$t(x)u_n(x) \geq \sum_{y \sim x} t(x, y) u_n(y).$$

Since the summation is over a finite number of terms, taking limits we conclude

$$t(x)u(x) \geq \sum_{y \sim x} t(x, y) u(y).$$

That is, u is superharmonic on E. □

Remark 1.2.2. In the classical potential theory, we should pay more attention when we consider the limit of superharmonic functions on an open set. If v_n is an increasing sequence of superharmonic functions on a domain ω, and if $v = \sup v_n$ is finite at one point in ω, then v is superharmonic on ω. But if $u_n \geq 0$ is a decreasing sequence of superharmonic functions on ω, then $u = \inf u_n$ need not be superharmonic on ω. However, there is a superharmonic function s on ω such that $s = u$ a.e. on ω. In fact, the second statement is a simplified version of the *convergence theorem* for superharmonic functions in the classical potential theory [27, pp. 74–78].

1.2.1 Examples of Superharmonic Functions on Networks

1. *The constants are the only superharmonic functions on a finite network X.* For, let s be a superharmonic function on X. Let z be a vertex in X such that $s(z) \leq s(x)$

for every x in X. Now $\Delta s(z) \leq 0$ which implies that
$0 \geq \Delta s(z) = \sum_{x_i \sim z} t(z, x_i)[s(x_i) - s(z)] \geq 0$, since $s(z) \leq s(x)$.
Consequently, $s(x_i) = s(z)$ for all $x_i \sim z$. Since X is connected, any x in X can be connected by a path $\{z, x_1, x_2, ..., x_n = x\}$. Then, the above argument shows that $s(x_1) = s(z)$, then $s(x_2) = s(x_1) = s(z)$ and so on. This permits us to conclude that finally $s(x) = s(z)$. We conclude therefore that s is a constant on X.

2. *Example of an infinite network in which every harmonic function is constant, but non-constant superharmonic functions exist.* Let $X = \{0, 1, 2, ...\}$ be an infinite network, with $t(0, 1) = 1$, $t(1, 0) = \frac{1}{2}$ and $t(n, n+1) = t(n+1, n) = \frac{1}{2}$ if $n \geq 1$.

   (a) Any harmonic function on X is a constant. For, let h be a harmonic function on X, and suppose $h(0) = a$. Then,

   $$0 = \Delta h(0) = t(0, 1)[h(1) - h(0)], \text{ so that } h(1) = h(0) = a. \text{ Again,}$$

   $$0 = \Delta h(1) = t(1, 0)[h(0) - h(1)] + t(1, 2)[h(2) - h(1)],$$
   $$\text{so that } h(2) = h(1) = a.$$

   Similar calculations show that $h(n) = a$ for all $n \geq 0$.

   (b) However, non-constant superharmonic functions exist on X. For example, set $s(n) = a - n\alpha, n \geq 0, \alpha > 0$. Then, $\Delta s(0) = -\alpha < 0$ and $\Delta s(n) = 0$ if $n \geq 1$. We say that s is a superharmonic function on X with *point harmonic support* at 0, since s is harmonic at every vertex except 0 and at the vertex 0, s is superharmonic but not harmonic.

3. *Example of an infinite tree with non-constant positive superharmonic functions.* Let T be a standard homogeneous tree of degree 3, that is every vertex has exactly 3 neighbours and the transition probability $p(x, y) = \frac{1}{3}$ if $x \sim y$. Let us fix a vertex e in T. For any vertex x in T, there is a unique path from e to x and let us denote the length of this path as $|x| = d(e, x)$. Note that for any vertex x in T, with $|x| = n \geq 1$, there is one vertex y with $|y| = n - 1$ and two vertices z with $|z| = n + 1$. Let $s(x) = 2^{1-|x|}$. Then,

   $$\Delta s(x) = \frac{1}{3}\left[2^{1-(n-1)} - 2^{1-n}\right] + \frac{2}{3}\left[2^{1-(n+1)} - 2^{1-n}\right] = 0,$$

   if $|x| = n \geq 1$, and

   $$\begin{aligned} \Delta s(e) &= \sum_{y \sim e} p(e, y)\left[s(y) - s(e)\right] \\ &= \sum_{y \sim e} \tfrac{1}{3}\left[1 - 2\right] = -1. \end{aligned}$$

Hence, s is a positive superharmonic function on T with point harmonic support at e.

4. *Example of a tree with non-constant positive harmonic functions.* Let $T = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$ be a tree, each vertex n having only two neighbours, with the transition probabilities $p(n, n+1) = \frac{3}{4}$ and $p(n, n-1) = \frac{1}{4}$ for any n. Then, $h(n) = 3^{-n}$ is a positive harmonic function on T. For,
$$\Delta h(n) = \tfrac{3}{4}\,[h(n+1) - h(n)] + \tfrac{1}{4}\,[h(n-1) - h(n)]$$
$$= \tfrac{3}{4}[3^{-n-1} - 3^{-n}] + \tfrac{1}{4}[3^{-n+1} - 3^{-n}] = 0.$$
Let $s(n) = 1$ if $n \leq 0$ and $s(n) = 3^{-n}$ if $n > 0$, that is $s = \inf(h(n), 1)$. Then, $\Delta s(0) = -\frac{1}{2}$ and $\Delta s(n) = 0$ if $n \neq 0$. Hence, s is a bounded positive superharmonic function with point harmonic support at the vertex 0.
5. *Example of a lattice in which non-constant positive superharmonic functions exist but every positive harmonic function is constant.* Let X be the set of lattice points in $\Re^3$, of the form (a, b, c) where a, b and c take the values $\ldots$, $-2, -1, 0, 1, 2, ...$ Take $t(x, y) = 1$ if $x \sim y$, otherwise $t(x, y) = 0$. Then, Courant constructs in the context of random walk and diffusion in a lattice [45, Sect. 3] the Green function $g(a, b, c) > 0$ such that $\Delta g(0, 0, 0) = -1$, $\Delta g(a, b, c) = 0$ otherwise, and $g(a, b, c) \to 0$ when $a^2 + b^2 + c^2 \to \infty$. Duffin obtains various asymptotic properties of $g(a, b, c)$ and proves that any positive harmonic function on X is a constant.

1.3 Green's Formulas

We shall prove now a version of the Green's formula in an infinite network and in a tree, motivated by a result of Duffin's in the above-mentioned discrete situation of lattice points in the Euclidean space $\Re^3$, of the form (a, b, c) where a, b and c take on the values ..., $-2, -1, 0, 1, 2$, ... For a real-valued function u on these lattice points, Duffin defines the Laplace operator D as
$$Du(a, b, c) = u(a+1, b, c) + u(a-1, b, c) + u(a, b+1, c) + u(a, b-1, c)$$
$$+ u(a, b, c+1) + u(a, b, c-1) - 6u(a, b, c).$$
Concerning the operator D, Duffin [45, p.233] remarks: *In physical problems the operator D is often used as an approximation to Δ. There are, however, some problems in which D appears directly, such as random walk and diffusion in a lattice. This concerns the motion of a particle which at each lattice point has an equal probability of jumping to a neighbouring lattice point. Another direct application arises if the lattice lines are regarded as metallic wires. This gives an infinite electrical network. If the electric potential of the lattice points is denoted by*

u, then at every insulated lattice point, $Du = 0$. At a source point, $Du = -w$ where w is the current entering this lattice point. These physical models are of suggestive value for the analysis of D.

Using this operator D, the classical Green's formula

$$\iint\limits_{\omega} (f\Delta g - g\Delta f)d\sigma = \int\limits_{\partial\omega} \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n} \right) ds$$

is shown to take the following form in the discrete case [45, Lemma 1]: Let u and v be lattice functions and let E be a finite set of lattice points. Then

$$\sum_E (uDv - vDu) = \sum_S [v(p)u(q) - u(p)v(q)].$$

Here $\sum\limits_E$ denotes the summation over E and $\sum\limits_S$ denotes the summation over the set S of points (p, q) where p and q are neighbours, p being in E and q being in the complement of E.

Such a formula in the discrete analysis on graphs [67] and in the framework of finite networks [18, Proposition 3.1] is known. Now, we have defined the Laplacian only for the interior vertices of E. A useful similar operator at the boundary vertices is the inner normal derivative [1]. If f is a function defined on a subset E of X, and if $\xi \in \partial E$, then the *inner normal derivative* of f at ξ is defined as

$$\frac{\partial}{\partial n^-} f(\xi) = \sum_{y \in E} t(\xi, y)[f(y) - f(\xi)].$$

If f is defined on X, and for a subset E, if $\xi \in \partial E$ then the *outer normal derivative* of f at ξ is defined as

$$\frac{\partial}{\partial n^+} f(\xi) = \sum_{y \notin E} t(\xi, y)[f(y) - f(\xi)].$$

Note that in the case of f being defined on X, if $\xi \in \partial E$, then

$$\Delta f(\xi) = \frac{\partial}{\partial n^+} f(\xi) + \frac{\partial}{\partial n^-} f(\xi).$$

Let E be a subset of X. Let us say that an edge L in X is an edge in E if and only if both the ends of L are vertices in E. Then, E can be considered as a graph, maybe not connected. Let us denote by $\Delta_E^\bullet$ the restriction of Δ to the subset E. That is, for any function f on E, $\Delta_E^\bullet f(x) = \Delta f(x)$ if $x \in \overset{0}{E}$, and $\Delta_E^\bullet f(\xi) = \frac{\partial}{\partial n^-} f(\xi)$ if $\xi \in \partial E$.

Proposition 1.3.1. *Let E be a finite subset of a network X with symmetric conductance . Let f be a function on E. Then,* $\sum_{x \in \overset{0}{E}} \Delta f(x) = -\sum_{\xi \in \partial E} \frac{\partial}{\partial n^-} f(\xi)$.

Proof. Note that $\sum_{z \in E} \Delta_E^{\bullet} f(z) = \sum_{x \in \overset{0}{E}} \Delta f(x) + \sum_{\xi \in \partial E} \Delta_E^{\bullet} f(\xi) = \sum_{x \in \overset{0}{E}} \Delta f(x) + \sum_{\xi \in \partial E} \frac{\partial}{\partial n^-} f(\xi)$. But $\sum_{z \in E} \Delta_E^{\bullet} f(z) = \sum_{y,z \in E} t(z,y)[f(y) - f(z)] = 0$, for in the second sum, corresponding to each term $t(z,y)[f(y) - f(z)]$ appearing in the sum, there is a term $t(y,z)[f(z) - f(y)]$ which cancels it out since $t(z,y) = t(y,z)$, with the assumption of the symmetric conductance. Hence we obtain the proposition. □

Note. If s is a harmonic function on a finite subset E, the expression $\sum_{\xi \in \partial E} \frac{\partial}{\partial n^-} s(\xi)$ is known as the *total inward flux of s on E*. Thus, the total inward flux of a harmonic function h on a finite set E should be 0. Given two functions f and g on a finite subset E of X, we write

$$(f,g)_E = \frac{1}{2} \sum_{x,y \in E} t(x,y)[f(x) - f(y)][g(x) - g(y)].$$

The following is a slight variant of the results given in Bendito et al. [18].

Theorem 1.3.2. *(Green's formula) Let f and g be two functions defined on a finite set E in a network X with symmetric conductance. Then,* $\sum_{x \in \overset{0}{E}} f(x) \Delta g(x) + (f,g)_E = -\sum_{\xi \in \partial E} f(\xi) \frac{\partial}{\partial n^-} g(\xi)$.

Proof. Extend f and g arbitrarily outside E. Then, for $x \in E$, $\Delta g(x) = \sum_{y \in X} t(x,y)[g(y) - g(x)]$ so that

$$\begin{aligned}
\sum_{x \in E} f(x) \Delta g(x) &= \sum_{x \in E} \sum_{y \in X} t(x,y) f(x)[g(y) - g(x)] \\
&= \sum_{x \in E} \sum_{y \in E} t(x,y) f(x)[g(y) - g(x)] \\
&\quad + \sum_{x \in E} \sum_{y \notin E} t(x,y) f(x)[g(y) - g(x)].
\end{aligned}$$

On the right side, in the first sum there are two terms $t(x,y)f(x)[g(y) - g(x)]$ and $t(y,x)f(y)[g(x) - g(y)]$ whose sum is $-t(x,y)[f(y) - f(x)][g(y) - g(x)]$; in the second sum, $t(x,y) > 0$ if and only if $x \in \partial E$ since $y \notin E$, so that it can be rewritten as

$$\sum_{\xi\in\partial E} f(\xi)\frac{\partial}{\partial n^+}g(\xi) = \sum_{\xi\in\partial E} f(\xi)[\Delta g(\xi) - \frac{\partial}{\partial n^-}g(\xi)].$$

Consequently, $\sum_{x\in \overset{0}{E}} f(x)\Delta g(x) = -(f,g)_E - \sum_{\xi\in\partial E} f(\xi)\frac{\partial}{\partial n^-}g(\xi)$. □

Corollary 1.3.3. *Let f and g be two functions defined on a finite set E in a network X with symmetric conductance. Then, $\sum_{\overset{0}{E}}[f\Delta g - g\Delta f] = -\sum_{\partial E}[f\frac{\partial g}{\partial n^-} - g\frac{\partial f}{\partial n^-}]$.*

Proof. Since $(f,g)_E = (g,f)_E$, the corollary follows from the above theorem. □

Remark 1.3.1. Suppose X is an infinite network with symmetric conductance. Then $\partial X = \phi$. The question is whether the above Green's Formula will yield the *dissipation formula* $(f,f) = -\sum_{x\in X} f(x)\Delta f(x)$. Kayano and Yamasaki [51] study the class of functions f for which this formula is satisfied.

1.4 Minimum Principle

Proposition 1.4.1. *Let E be circled and $\overset{0}{E}$ be connected. If s is a superharmonic function on E that attains its minimum at a vertex in $\overset{0}{E}$, then s is constant.*

Proof. Let $\alpha = \inf_E s(x)$. By hypothesis, there is a vertex $y \in \overset{0}{E}$ such that $s(y) = \alpha$. Let $z \in \overset{0}{E}$ be an arbitrary vertex. Since $\overset{0}{E}$ is connected, there is a path $\{y, z_1, ..., z_n = z\}$ connecting y and z. Since $0 \geq \Delta s(y) = \sum t(y, y_i)[s(y_i) - s(y)]$, and $s(y)$ is the minimum value, we conclude that $s(y_i) = s(y) = \alpha$ for every $y_i \sim y$. In particular, $s(z_1) = \alpha$. Proceeding step-by-step, we arrive at the value $s(z) = \alpha$. Since $z \in \overset{0}{E}$ is an arbitrary vertex, $s(x) = \alpha$ for all $x \in \overset{0}{E}$.

Let $\xi \in \partial E$. Since E is circled, for some $x \in \overset{0}{E}$, $x \sim \xi$. Hence, by the earlier remarks, $s(x) = \alpha$ and $s(\xi) = \alpha$. Consequently, $s \equiv \alpha$ on E. □

Remark 1.4.1. In particular, if $s \geq 0$ is superharmonic on X and if $s(x) = 0$ at a vertex x in X, then $s \equiv 0$.

A variation of the above Proposition 1.4.1 is the following.

Theorem 1.4.2. *(Minimum Principle) Let E be an arbitrary proper subset of X. Let s be a superharmonic function on E, attaining its minimum on E. Then $\inf_{\partial E} s = \inf_E s$.*

Proof. Let $\alpha = \inf_{\partial E} s$ and $\beta = \inf_{E} s$. Then, $\alpha \geq \beta > -\infty$. Suppose $\alpha > \beta$. Then $s(z) = \beta$ for some $z \in \overset{0}{E}$, by hypothesis. Choose $y \notin$E. There is a path $\{z = x_0, x_1, ..., x_n = y\}$ connecting z and y. Let i be such that $x_k \in \overset{0}{E}$ for all $k \leq i$ and $x_{i+1} \notin \overset{0}{E}$. Then $i < n$.

Now, $\beta t(z) = t(z)s(z) \geq \sum_{z_i \sim z} t(z, z_i)s(z_i) \geq \beta \sum t(z, z_i) = \beta t(z)$.

It is clear then $s(z_j) = \beta$ for every $z_j \sim z$. In particular $s(x_1) = \beta$. Continuing this process, we see that $s(x_{i+1}) = \beta$. Now $x_{i+1} \notin \overset{0}{E}$, but $x_{i+1} \sim x_i \in \overset{0}{E}$. Hence $x_{i+1} \in \partial E$. Consequently, $\inf_{x \in \partial E} s(x) \leq \beta$, which is a contradiction. This proves $\alpha = \beta$. □

Corollary 1.4.3. *Let E be a finite subset of X. Let u be superharmonic on E and h be subharmonic on E such that $u \geq h$ on ∂E. Then, $u \geq h$ on E.*

Proof. Let $s = u - h$ on E. Then, s is a superharmonic function on E such that $s \geq 0$ on ∂E. Since E is a finite set, s attains its minimum value on E. Hence, by the Minimum Principle, $s \geq 0$ on E; that is, $u \geq h$ on E. □

The above corollary is very useful on many occasions in the following forms:

(a) Let E be a finite subset of X. Let u be a superharmonic function on E. Then, for any x in E, $u(x) \geq \inf_{z \in \partial E} u(z)$.
(b) Let h_1, h_2 be two harmonic functions on a finite subset E of X. If $h_1 = h_2$ on ∂E, then $h_1 = h_2$ on E.
(c) Let h be a harmonic function defined on a finite subset E of X. Then for any $x \in E$, $\inf_{z \in \partial E} h(z) \leq h(x) \leq \sup_{z \in \partial E} h(z)$.

However, for the above assertions, the assumption that E is a finite set is important. For, consider the following example of an infinite subset E where the minimum principle in the form stated above is not valid:
Let $X = \{0, 1, 2, 3, ...\}$ and $E = \{1, 2, 3, ...\}$ with $t(n, n+1) = a$ if $n \geq 0$, $t(n, n-1) = b$ if $n \geq 1$, $a + b = 1$, and $a > b$. Then, $\overset{0}{E} = \{2, 3, 4, ...\}$ and $\partial E = \{1\}$. Let $h(n) = \left(\frac{b}{a}\right)^n$ for $n \geq 0$. Then, for $n \geq 2$,

$$\begin{aligned} ah(n+1) + bh(n-1) &= a\left(\frac{b}{a}\right)^{n+1} + b\left(\frac{b}{a}\right)^{n-1} \\ &= \left(\frac{b}{a}\right)^n \left[a.\frac{b}{a} + b.\frac{a}{b}\right] \\ &= \left(\frac{b}{a}\right)^n [b + a] = \left(\frac{b}{a}\right)^n \\ &= h(n). \end{aligned}$$

Hence, h is harmonic on E and bounded also. But $h(x) \leq h(1) = \inf_{z \in \partial E} h(z)$ for all x in E.

It is possible however to obtain useful modified minimum principles as given below. (Remark that corresponding to the minimum principle for superharmonic functions, we can prove a maximum principle for subharmonic functions. This is because a function u is subharmonic if and only if $-u$ is superharmonic.)

Proposition 1.4.4. *Assume that there exists a function $p > 0$ on X such that for any superharmonic function s on X, if $p + s \geq 0$ on X, then $s \geq 0$ on X. Let u be a subharmonic function on an arbitrary proper subset E of X such that $u \leq p$ on E. If $u \leq 0$ on ∂E, then $u \leq 0$ on E.*

Proof. Let $v = \sup\{u, 0\}$. Then, v is subharmonic on E and $v = 0$ on ∂E. If v is thought of as a function defined on X by taking values 0 outside E, then v is subharmonic on X and $v \leq p$ on X. Since $p - v \geq 0$ on X, by the assumption $-v \geq 0$, that is $v \leq 0$ on X and hence $u \leq 0$ on E. □

Proposition 1.4.5. *Suppose every positive superharmonic function on X is a constant. Let s be a lower bounded superharmonic function on an arbitrary proper subset E of X such that $s \geq m$ on ∂E. Then, $s \geq m$ on E.*

Proof. Suppose $s \geq \lambda$ on E. Let $v = \inf\{s, m\}$. Then, v is superharmonic on E and $v = m$ on ∂E. By giving values m outside E, we shall assume that v is defined on X. Then, v is superharmonic and also lower bounded on X. Hence by the assumption, v is a constant which should be m. This implies that $s \geq m$ on E. □

Example of a tree in which any positive superharmonic function is a constant: Let T be a homogeneous tree, each vertex having exactly 3 neighbours. Fix a vertex e and for $x \in T$, write $|x| = d(e, x) =$ the length of the unique path joining e to x. Note that for any $n \geq 1$, if $E_n = \{x : |x| \leq n\}$, then $\overset{0}{E_n} = E_{n-1}$ and $\partial E_n = \{x : |x| = n\}$. For $|x| = n \geq 1$, define $p(x, y) = \frac{1}{2}$ if $y \sim x$ and $|y| = n-1$ and $p(x, y) = \frac{1}{4}$ if $y \sim x$ and $|y| = n + 1$; define $p(e, y) = \frac{1}{3}$ for each $y \sim e$. Let $h(x) = |x|$. It is easy to check that $\Delta h(e) = 1$ and $\Delta h(x) = 0$ if $x \neq e$.

Suppose $s > 0$ is a superharmonic function on T. Let $\alpha = \inf_{|x| \leq m} s(x)$, so that for some z, $|z| \leq m$, $\alpha = s(z) \leq s(x)$ for all $x \in E_m$. Let y be an arbitrary vertex in T such that $|y| > m$. Let n be a large integer, $n > |y|$. Define $h_n(x) = \alpha \frac{n-|x|}{n-m}$, which is harmonic at every $x \neq e$. Note then on ∂E_m, $s(x) \geq \alpha = h_n(x)$ and on ∂E_n, $s(x) > 0 = h_n(x)$. Then, by the Minimum Principle (Corollary 1.4.3), $s(x) \geq h_n(x)$ on $E_n \setminus \overset{0}{E_m}$. In particular, $s(y) \geq h_n(y)$. Allow $n \to \infty$ to conclude that $s(y) \geq \alpha$. As a consequence, we conclude that $s(x) \geq \alpha$ for all $x \in T$. But $s(z) = \alpha$, that is the superharmonic function attains its minimum on T. Hence s is a constant (Proposition 1.4.1).

1.5 Infinite Trees

A network need not have a symmetric conductance as in the case of the important example of random walks where the probability of transition $p(x, y)$ from a state x to a state y may not be the same as the probability of transition $p(y, x)$ from y to x. In such cases, defining $t(x, y) = \frac{1}{2}[p(x, y) + p(y, x)]$, we can obtain some useful results. In some other occasions, the network may be a tree, in which case a different method of obtaining symmetric conductance can be used.

An infinite tree T is a connected, locally finite, infinite graph without self-loops; moreover, if x and y are neighbours, there is no path $\{x = s_0, s_1, ..., s_n = y\}, n \geq 2$, with distinct vertices such that $s_i \sim s_{i+1}$ for $0 \leq i \leq n-1$. On T, it is assumed that transition probabilities $p(x, y) \geq 0$ are given such that $p(x, y) > 0$ if and only if x and y are neighbours and $\sum_{y \sim x} p(x, y) = 1$ for any x in T.

T is said to be *a standard homogeneous tree* of degree $q + 1$, $q \geq 2$, if each vertex has exactly $(q + 1)$ neighbours and $p(x, y) = (q + 1)^{-1}$ if $x \sim y$. Analogously, we can think of a standard homogeneous tree of degree 2 as follows: $T = \{..., x_{-2}, x_{-1}, x_0, x_1, x_2, ...\}$, each vertex x_i having only two neighbours x_{i-1} and x_{i+1}, and $p(x_i, x_{i+1}) = p(x_i, x_{i-1}) = \frac{1}{2}$ for every x_i. (In the study of superharmonic functions on a standard homogeneous tree T of degree $q + 1$, there will be many differences when the order $q = 1$ and $q \geq 2$.) Fix a vertex e in T and for a vertex x in T, denote $|x| = d(e, x)$, the distance of x measured from e. That is, if $\{e, x_1, x_2, ..., x_n = x\}$ is the (geodesic) path connecting x to e, then $|x| = d(e, x) = n$. Define the function s on T such that $s(e) = 1$ and $s(x) = q^{-i}$ if $|x| = i \geq 1$. Note that if $|x| = i$, then x has only one neighbour y with $|y| = i-1$ and q neighbours $z_j, 1 \leq j \leq q$, such that $|z_j| = i+1$. In a standard homogeneous tree T, it is easy to assert the existence of many non-proportional harmonic functions on T, with the help of the following proposition.

Proposition 1.5.1. *Let e be a fixed vertex in a standard homogeneous tree T of degree $q + 1$, $q \geq 1$, and write $|x| = d(e, x)$ for a vertex x in T. Let $u(x)$ be a function defined when $|x| = n - 1$ and $|x| = n$, for a fixed $n \geq 1$. Then, there exists a function $v(x)$ defined for $|x| \geq n - 1$ such that $v(x) = u(x)$ if $|x| = n - 1$ and $|x| = n$; and $\Delta v(x) = 0$ if $|x| \geq n$.*

Proof. Let $|z| = n$. Then, z has q neighbours $y_1, y_2, ..., y_q$ such that $|y_j| = n + 1$ if $1 \leq j \leq q$ and one neighbour y_0 such that $|y_0| = n - 1$. Take $\lambda = \frac{1}{q}[(q + 1)u(z) - u(y_0)]$ and set $v(y_j) = \lambda$,$1 \leq j \leq q$, $v(y_0) = u(y_0)$ and $v(z) = u(z)$ so that $\Delta v(z) = 0$. Similarly, extend the definition of $v(x)$ for all $|x| = n + 1$, so that $\Delta v(x) = 0$ if $|x| = n$ and $v(x) = u(x)$ if $|x| = n - 1$ or n. An analogous procedure yields a function $v(x)$ for all x, $|x| \geq n - 1$, such that $v(x) = u(x)$ if $|x| = n - 1$ or n, and $\Delta v(x) = 0$ if $|x| \geq n$. □

Corollary 1.5.2. *Let e be a fixed vertex in a standard homogeneous tree T of degree $q+1$, $q \geq 1$. Let $u(x)$ be a harmonic function defined on $E = \{x : |x| \leq n, n \geq 1\}$. Then, there exists a harmonic function v on T such that $v = u$ on E.*

Proof. Extend u as in the theorem to find a function v such that $v = u$ on E and $\Delta v(x) = 0$ if $|x| \geq n$. Since by the assumption $\Delta u(x) = 0$ if $|x| \leq n-1$, we conclude that v is harmonic on T and $v = u$ on E. □

Corollary 1.5.3. *For any vertex z in a standard homogeneous tree T of degree $q+1$, $q \geq 1$, there exists a superharmonic function $q_z(x)$ on T such that $\Delta q_z(x) = -\delta_z(x)$ for all x in T.*

Proof. Measuring distances from z, define a function $u(x)$ on $E = \{x : |x| \leq 1\}$ such that $u(z) = 2$ and $u(y) = 1$ if $y \sim z$. Then, by the above theorem, there exists a function v on T such that $\Delta v(x) = 0$ if $|x| \geq 1$ and $v(x) = u(x)$ if $|x| \leq 1$. Since, $\Delta v(z) = \sum_{y \sim z} (q+1)^{-1} [v(y) - v(z)] = \sum_{y \sim z} (q+1)^{-1} [u(y) - u(x)] = \sum_{y \sim z} (q+1)^{-1} [1-2] = -1$, we conclude that $\Delta v(x) = -\delta_z(x)$ for all x in T. □

Remark 1.5.1. 1. Actually, given any vertex e in a standard homogeneous tree of degree $q+1$, $q \geq 2$, we can exhibit a positive superharmonic function s on T such that $\Delta s(x) = -\delta_e(x)$ for every x in T. For, define the function s on T such that $s(e) = 1$ and $s(x) = q^{-i}$ if $|x| = i \geq 1$. Note that if $|x| = i$, then x has only one neighbour y with $|y| = i-1$ and q neighbours z_j, $1 \leq j \leq q$, such that $|z_j| = i+1$. Consequently, if $|x| = i \geq 1$, then

$$\Delta s(x) = (q+1)^{-1}[q^{-i+1} - q^{-i}] + q(q+1)^{-1}[q^{-i-1} - q^{-i}] = 0, \text{ and}$$

$$\Delta s(e) = (q+1)(q+1)^{-1}(q^{-1} - 1) = (q^{-1} - 1) < 0.$$

Hence, $G_e(x) = \frac{s(x)}{(1-q^{-1})}$ is a positive function on the standard homogeneous tree T such that $\Delta G_e(x) = -\delta_e(x)$ for all x in T.

2. *Bounded harmonic functions on a standard homogeneous tree of degree $q+1$, $q \geq 2$.* Let T be a homogeneous tree of degree $q+1$, $q \geq 2$. Fix a vertex e and one of its neighbours a. Let $E = \{x : \text{the path joining } x \text{ to } e \text{ passes through } a\}$. Then, $a \in E$ and $e \notin E$. Denote by $d(x,y)$ the distance between two vertices x and y. Define a function u on T such that $u(x) = q+1-q^{-d(e,x)}$ if $x \notin E$ and $u(x) = q^{-d(a,x)}$ if $x \in E$. Then, $u(e) = q$ and $u(a) = 1$.

$$\begin{aligned}\Delta u(e) &= q(q+1)^{-1}\left[(q+1-q^{-1}) - q\right] + (q+1)^{-1}[1-q] \\ &= (q+1)^{-1}[(q-1) + (1-q)] = 0, \text{ and}\end{aligned}$$

$$\Delta u(a) = q(q+1)^{-1}\left[q^{-1} - 1\right] + (q+1)^{-1}[q-1] = 0.$$

$$\begin{aligned}\Delta u(x) &= q(q+1)^{-1}\left[q^{-n-1} - q^{-n}\right] + (q+1)^{-1}\left[q^{-n+1} - q^{-n}\right] \\ &= 0, \text{ if } x \in E \text{ and } d(a,x) = n.\end{aligned}$$

$$\begin{aligned}\Delta u(x) &= q(q+1)^{-1}\left[(q+1-q^{-n-1})-(q+1-q^{-n})\right] \\ &\quad +(q+1)^{-1}\left[(q+1-q^{-n+1})-(q+1-q^{-n})\right] \\ &= 0, \text{ if } x \notin E \text{ and } d(e,x)=n.\end{aligned}$$

Hence, $u(x)$is a non-constant bounded positive harmonic function on T. Since there are $(q+1)$ neighbours for e, there are at least $(q+1)$ non-proportional bounded positive harmonic functions on T.

3. Let T be a standard homogeneous tree of degree $q+1$, $q \geq 2$. Let $u=0$ on $E=\{x:|x|\leq n, n\geq 1\}$. If $z \in \partial E=\{x:|x|=n\}$, let $y_1, y_2, ..., y_q$ be the q neighbours of z such that $|y_j|=n+1$, $1\leq j\leq q$. Let $v(y_j)=\lambda_j$, where all λ_j are not 0 but $\sum_1^q \lambda_j=0$, and $v=0$ on E. Then, $\Delta v(z)=0$. Proceeding with similar constructions as in the proof of Proposition 1.5.1, we arrive at a harmonic function v on T such that v is not identically 0 but $v=0$ on E. Note that this is in contrast to the situation in the classical potential theory where if a harmonic function h on a domain ω in the Euclidean space $\Re^n, n\geq 2$, is 0 in a neighbourhood of a point in ω, then $h\equiv 0$ on ω. However, note that if $T^\bullet$ is a standard homogeneous tree of degree 2 and if $h^\bullet$ is a harmonic function on $T^\bullet$ vanishing at two consecutive vertices (in particular, if $h^\bullet=0$ in a neighbourhood of a vertex in $T^\bullet$), then $h^\bullet\equiv 0$ on $T^\bullet$.

   In fact, on a standard homogeneous tree T degree 2, if h is a function such that $\Delta h(x)=0$ for every $x\in T$, then h is determined by its values at two adjacent vertices and is of the form $h(x_i)=a+i(b-a)$ for all i, where a and b are constants. For, suppose $\Delta h\equiv 0$. Let $h(x_0)=a$ and $h(x_1)=b$. Then, to find the value $h(x_2)$ for example, use the condition

$$0=\Delta h(x_1)=\frac{1}{2}[h(x_2)-h(x_1)]+\frac{1}{2}[h(x_0)-h(x_1)]$$

   to conclude that $h(x_2)=2h(x_1)-h(x_0)=a+2(b-a)$. Similar calculations give the values of $h(x_3), h(x_4)..., h(x_{-1}), h(x_{-2})...$ to arrive at the conclusion $h(x_i)=a+i(b-a)$, for all i. Consequently, the only positive harmonic functions in this tree are the positive constants and every bounded harmonic function is a constant.

   Actually, in the case of a standard homogeneous tree of degree 2, it is not possible to find a non-constant positive function s on T such that $\Delta s(x)\leq 0$ for all x. For, let $s\geq 0, \Delta s\leq 0$ on T. Since s is a non-negative superharmonic function on T, if $s=0$ at a vertex in T, then $s\equiv 0$ on T by the Minimum Principle. Hence, if $\min\{s(x_0), s(x_1), s(x_{-1})\}=a$, then $a>0$. For an integer $m>1$, let $h_m(x)=a\frac{m-|x|}{m-1}$, measuring distances from x_0. Then, $\Delta h_m(x)=0$ if $x\neq x_0$. Then, on the segment $[x_1, x_m]$,

$$s(x_1) \geq a \ = h_m(x_1) \text{ and}$$

$$s(x_m) > 0 = h_m(x_m).$$

Hence, by Corollary 1.4.3, $s(x) \geq h_m(x)$ on $[x_1, x_m]$. Take any vertex z, $|z| > 1$. Then, $s(z) \geq a\frac{m-|z|}{m-1}$ for any $m > |z|$. Allow $m \to \infty$ to conclude that $s(z) \geq a$. Consequently, $s(x) \geq a$ if x is any vertex in $\{x_0, x_1, x_2, ...\}$. Similarly for any vertex x in $\{..., x_{-2}, x_{-1}, x_0\}$ also, $s(x) \geq a$. This means that the superharmonic function s on T attains its minimum value a at a vertex in T. Hence $s(x) = a$ for all x in T.

Remark 1.5.2. The above results, concerning the existence of positive superharmonic functions on standard homogeneous trees, bring out some of the differences between standard homogeneous trees of degree $q+1$ when $q \geq 2$ and $q = 1$. These are akin to the differences in the study of classical potential theory on $\Re^n, n \geq 2$. For example, $|x|^{2-n}$ is a positive superharmonic function on $\Re^n, n \geq 3$. But any positive superharmonic function on $\Re^2$ is a constant; this assertion [27, p.28] is a generalised version of the classical Liouville theorem in the complex plane.

In the case of homogeneous trees, there are no terminal vertices. One simple example of a tree with terminal vertices is $T = \{x_0, x_1, x_2, ...\}$ where x_0 is a terminal vertex and every vertex $x_i, i \geq 1$, has only two neighbours x_{i-1} and x_{i+1}; if $i \geq 1$, then $p(x_i, x_{i+1}) = p_i$ and $p(x_i, x_{i-1}) = q_i$ such that $p_i + q_i = 1$ and $p(x_0, x_1) = 1$. In this tree, constants are the only harmonic functions on T. For, if h is harmonic on T and $h(x_0) = a$, then $0 = \Delta h(x_0) = h(x_1) - h(x_0)$ so that $h(x_1) = a$. The values of h at other vertices are calculated as in the previous example to show that $h(x) = a$ for all x.

Now, to obtain the Green's formulas on a network, we have used the fact that $t(x, y)$ is symmetric. To obtain similar formulas in a tree T, we shall introduce the following modifications. Fix a vertex e in T. For a vertex x, let $\{e, x_1, ..., x_n, x\}$ be the path joining e and x. Let $\varphi(x) = \frac{p(e,x_1)p(x_1,x_2)...p(x_n,x)}{p(x,x_n)p(x_n,x_{n-1})...p(x_1,e)}$; take $\varphi(e) = 1$. Note that if x and y are neighbours, then $\varphi(y) = \frac{p(x,y)}{p(y,x)}\varphi(x)$; that is, $\varphi(y)p(y, x) = \varphi(x)p(x, y)$. Clearly, this equality holds even when x and y are not neighbours. Thus, for any pair of vertices x and y in T, if we define $\psi(x, y) = \varphi(x)p(x, y)$, then $\psi(x, y) = \psi(y, x) \geq 0$, and $\psi(x, y) = 0$ if and only if x and y are not neighbours.

For any real function u on T, we use the expression

$$\Delta u(x) = \sum_{y \sim x} p(x, y)[u(y) - u(x)].$$

Let us write $\Delta^\bullet u(x) = \sum\limits_{y \sim x} \psi(x, y)[u(y) - u(x)]$. Then,

$$\Delta^\bullet u(x) = \sum_{y \sim x} \varphi(x)p(x, y)[u(y) - u(x)] = \varphi(x)\Delta u(x).$$

Similarly, if E is any proper subset of T and if $s \in \partial E$, then write

$$\frac{\partial^\bullet}{\partial n^-} u(s) = \sum_{y \sim s, y \in E} \psi(s, y)[u(y) - u(s)],$$

so that $\frac{\partial^\bullet}{\partial n^-} u(s) = \varphi(s) \frac{\partial}{\partial n^-} u(s)$. Write

$$(u, v)^\bullet_E = \frac{1}{2} \sum_{x, y \in E} \psi(x, y)[u(x) - u(y)][v(x) - v(y)].$$

Theorem 1.5.4. *(Green's formulas in trees) Let f and g be two real-valued functions on a tree T. Let E be a finite subset of T. Then,*

i. $\sum_{x \in \overset{0}{E}} \varphi(x) f(x) \Delta g(x) + (f, g)^\bullet_E = - \sum_{s \in \partial E} \varphi(s) f(s) \dfrac{\partial}{\partial n^-} g(s).$

ii. $\sum_{x \in \overset{0}{E}} \varphi(x)[f(x) \Delta g(x) - g(x) \Delta f(x)] = - \sum_{s \in \partial E} \varphi(s)[f(s) \dfrac{\partial}{\partial n^-} g(s) - g(s) \dfrac{\partial}{\partial n^-} f(s)].$

Proof. Let us consider T as an infinite network with the symmetric conductance $\psi(x, y)$. Then, by Theorem 1.3.2, $\sum_{x \in \overset{0}{E}} f(x) \Delta^\bullet g(x) + (f, g)^\bullet = - \sum_{s \in \partial E} f(s) \frac{\partial^\bullet}{\partial n^-} g(s)$. Rewriting this, we get *(i)* and then *(ii)* since $(f, g)^\bullet_E = (g, f)^\bullet_E$. □

Remark 1.5.3. Suppose T is a standard homogeneous tree of degree $q + 1$. Fix a vertex e and measure distances from e. Let $B_m = \{x : |x| \leq m\}$. Then, $\{x : |x| < m\} = \overset{\circ}{B}_m$ and $\partial B_m = \{x : |x| = m\}$. Since $p(x, y) = (q + 1)^{-1}$ for any pair of neighbours x and y, we find $\varphi \equiv 1$. Hence, for functions f and g defined on B_m, we can write $\sum_{|x| < m} [f(x) \Delta g(x) - g(x) \Delta f(x)] = - \sum_{|s| = m} [f(s) \dfrac{\partial}{\partial n^-} g(s) - g(s) \dfrac{\partial}{\partial n^-} f(s)]$.

Chapter 2
Potential Theory on Finite Networks

Abstract Solving Kirchhoff's problem in a finite electrical network opens up various possibilities for the study of functions in networks. A method to solve the Dirichlet-Poisson equation in a finite network, by using algebraic techniques involving the inverses of submatrices of the Laplace matrix is given here; the novelty is the Laplace matrix need not be symmetric. The same method is used to explain Equilibrium Principle, Condenser Principle and Balayage. Later, assuming the Laplace matrix is symmetric, the Dirichlet semi-norm is defined and the Dirichlet Principle is proved. This procedure can be modified to develop a potential theory associated with the Schrödinger operators in a finite network. However, another approach based on the properties of Schrödinger superharmonic functions is adopted here in order to bring into focus the potential-theoretic methods that will be used later in the context of infinite networks.

2.1 Incidence Matrix, Kirchhoff's Problem

Minimum Principle, Domination Principle, Balayage, Capacities and Equilibrium Principle are some of the important concepts in the study of classical potential theory on Euclidean spaces. These are directly related to the three problems associated with electrostatics and Newtonian attraction in $\Re^3$ considered by Gauss in 1840: They are the Equilibrium Problem (known also as the Robin problem), the Balayage Problem (so known by the method of solution given by Poincaré) and the Dirichlet Problem (name given by Riemann). In an attempt to understand the interconnections among these notions, the remarkable paper Choquet et Deny [33] investigates them in the context of finite discrete sets. Inspired by this paper, this chapter is devoted to the study of potential theory on finite networks.

Let X be a connected, directed finite graph, with an arbitrary orientation, without loops, having n vertices and m edges. Let $x \sim y$ denote that there is an edge $e = [x, y]$ connecting x and y. Let D be its associated *incidence matrix* with n rows and m columns, that is, if we denote the vertices by x_i and the edges by e_j, then the

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incidence matrix $D = (d_{ij})$ is such that $d_{ij} = 1$ if x_i is the head $\alpha(e_j)$ of the edge e_j, $d_{ij} = -1$ if x_i is the tail $\beta(e_j)$ of e_j, and $d_{ij} = 0$ if x_i is not an end vertex of e_j. Let C_0 (respectively C_1) denote the class of real-valued functions defined on the set of vertices (respectively edges). Considering C_0 and C_1 as column vectors, we get the mappings $D : C_1 \to C_0$ and $D^t : C_0 \to C_1$ where D^t denotes the transpose of D. If $\varphi \in C_1$, then $D\varphi(x) = \sum_{x\in\alpha(e)} \varphi(e) - \sum_{x\in\beta(e)} \varphi(e)$. If $u \in C_0$ and if $e = [x, y]$ is an edge, x being the head of the edge and y being the tail of the edge, then $D^t u(e) = u(x) - u(y)$. Hence, if $u \in \ker D^t$, then $u(a) = u(b)$ for any two neighbouring vertices a and b. Since X is connected, this implies that if $u \in \ker D^t$, then u is a constant, so that $\ker D^t$ has dimension 1. Hence $\dim(ImD^t) = n - 1$. Let Z denote $\ker D$ (that is $\varphi \in Z$ if and only if for any vertex $x \in X$, we have $\sum_{\alpha(e)=x} \varphi(e) = \sum_{\beta(e)=x} \varphi(e)$ in which case φ is called a *flow*). Then, by considering the inner-product defined on C_1, let us denote by B the orthogonal complement of Z so that we can write $C_1 = Z \oplus B$. Now, $\dim B = \dim(ImD) =$ rank of $D =$ rank of $D^t = \dim(ImD^t) = n - 1$, so that $\dim Z$ is $m - (n - 1)$. Moreover, if $f \in ImD^t$, that is $f = D^t\varphi$, and if g is arbitrary in Z, then $(g, f) = (g, D^t\varphi) = (Dg, \varphi) = (0, \varphi) = 0$, so that $ImD^t \subset B$. Since $\dim(ImD^t) = n - 1 = \dim B$ also, we conclude that $B = ImD^t$.

Let us consider an electrical network with $\gamma(e)$ representing the conductance on the edge e. Then, the *Kirchhoff's problem* can be stated as follows [24]: Given an external current c, find the internal current i and the voltage w satisfying the Ohm's law and the Kirchhoff's current and voltage laws. When we state this problem by using the above notations, it reads as follows: Given $c \in C_1$, find i and w in C_1 such that $c - i \in Z$, $w \in B$, and $i(e) = \gamma(e)w(e)$ for every edge e. Let us denote by T the diagonal $m \times m$ matrix whose diagonal entries are $\gamma(e)$. Then $i = Tw = T(D^t\varphi)$ for some $\varphi \in C_0$ since $B = ImD^t$. Also, the condition $c - i \in Z$ means that $Dc = Di = DT(D^t\varphi)$. Hence, the Kirchhoff's problem is: Given $c \in C_1$, find $\varphi \in C_o$ such that $Dc = DTD^t\varphi$.

Now $DTD^t\varphi(x) = 0$ expands as $\sum_{y\sim x} t(x, y)[\varphi(y) - \varphi(x)] = 0$, where $t(x, y)$ is the symmetric conductance $\gamma(e)$ on the edge e joining x and y. If we write $t(x) = \sum_{y\sim x} t(x, y)$, then $t(x) > 0$ for every vertex x and $\varphi(x) = \sum_{y\sim x} \frac{t(x,y)}{t(x)}\varphi(y)$ expresses the mean-value property of φ. In keeping with the notation used earlier, let us write $-\Delta = DTD^t$. Thus, if $\varphi \in C_0$, then $\Delta\varphi \in C_0$ where $\Delta\varphi(x) = \sum_{y\sim x} t(x, y)[\varphi(y) - \varphi(x)]$ for each vertex x. That is, if we denote the vertices as $x_1, x_2, ..., x_n$, then $\Delta = (c_{ij})$ is a symmetric $n \times n$ matrix, $c_{ii} = -\sum_{j\neq i} t(x_i, x_j)$ for every i, $c_{ij} = t(x_i, x_j) \geq 0$ if $i \neq j$, $\sum_j c_{ij} = \sum_i c_{ji} = 0$; further, since rank of $D =$ rank of $D^t = n - 1$, and since T is a non-singular matrix, the rank of Δ is $n - 1$; since $t(x, y) = t(y, x)$, we also find that $\sum_x \Delta\varphi(x) = 0$. Thus, for a given $f \in C_0$, if we can find some $\varphi \in C_0$ such that $\Delta\varphi = f$, then necessarily $\sum_x f(x) = 0$. We shall prove that this condition on f is sufficient to find the solution φ in the equation $\Delta\varphi = f$. Of course, φ will be determined only up to an additive constant, since $\ker \Delta$ is the set of constant functions in C_0. In particular, the Kirchhoff's problem $(-\Delta)\varphi = DTD^t\varphi = Dc$ has a solution (Theorem 2.2.7) for a given $c \in C_1$ since

$$\sum_x Dc(x) = \sum_x \left[\sum_{\alpha(e)=x} c(e) - \sum_{\beta(e)=x} c(e) \right] = 0.$$

2.2 Dirichlet-Poisson Equations in Finite Networks

In this section, we give a method to obtain the solution of the Dirichlet-Poisson equation on a subset F of X, by using algebraic methods (essentially inverses of sub-matrices of the Laplacian matrix) rather than Green functions expressed by means of equilibrium measures as in Bendito et al. [18] and [19] which assume symmetric conductance. Here the Laplacian matrix $\Delta = (c_{ij})$ is defined such that $c_{ij} = t(x_i, x_j)$ may not be the same as $c_{ji} = t(x_j, x_i)$ unlike in the special case of an electric network, that is Δ is generally assumed to be a non-symmetric matrix. But $c_{ii} = -\sum_{j \neq i} t(x_i, x_j)$ so that $\sum_j c_{ij} = 0$ for every i. The *classical Dirichlet problem* is as follows: Suppose F is a proper subset of X and f is a function given on ∂F. Then, is it possible to find a function u defined on F such that $\Delta u = 0$ on $\overset{0}{F}$ and $u = f$ on ∂F? If $X = \Re^n, n \geq 2$, a similar problem in the classical potential theory does not always have a solution, even if F is a bounded set in $\Re^n$. However, in a finite network, it has a solution. Actually, we prove a more general result: Suppose F is a subset of X and let $E \subseteq \overset{0}{F}$. If f is a function on $F \backslash E$, then there exists a unique u on F such that $\Delta u = 0$ on E and $u = f$ on $F \backslash E$. The interest in this problem comes from the fact that many potential-theoretic problems can be posed as Dirichlet problems, as we shall see presently. Examples of the Dirichlet problem arising in some other contexts are given in Doyle and Snell [44], Tetali [64] and Biggs [24]:

(a) Consider the following problem in *random walks*: Let X be the state space provided with the transition probabilities $p(x, y)$. Given two vertices a and b in X, what is the probability of the walker reaching a before b?

To solve this, let us denote by $\varphi(x)$ the probability of the walker starting at the vertex x and reaching a before b. Then $\varphi(a) = 1$ and $\varphi(b) = 0$. If x is different from a and b, then the walker starting at x takes a first step to a neighbouring vertex y and then from y he should reach a before reaching b. Consequently, $\varphi(x) = \sum_y p(x, y)\varphi(y)$. Since $p(x, y)$ are probabilities, $\sum_y p(x, y) = 1$ for any fixed x. Hence, $\sum_y p(x, y)\varphi(x) = \varphi(x) = \sum_y p(x, y)\varphi(y)$, that is,

$\sum_y p(x, y)\,[\varphi(y) - \varphi(x)] = 0$; hence, $\Delta\varphi(x) = 0$ if x is different from a and b. Thus, the probability $\varphi(x)$ is the solution to the Dirichlet problem with $F = X$, $E = X \setminus \{a, b\}$, $\varphi(a) = 1$ and $\varphi(b) = 0$.

(b) In the above random walk, what is the *escape probability* $p_{esc}(a, b)$ which is defined as the probability of starting at a and visiting b before returning to a?
As above, let $\varphi(x)$ represent the probability of the walker starting at x and visiting a before reaching b. Let λ represent the walker leaving a and arriving at a without visiting b, so that $1 - \lambda = p_{esc}(a, b)$. Since $\lambda = \sum_y p(a, y)\varphi(y)$, we have

$$\begin{aligned} p_{esc}(a, b) &= 1 - \lambda \\ &= \varphi(a) - \sum_y p(a, y)\varphi(y) \\ &= -\sum_y p(a, y)\,[\varphi(y) - \varphi(a)] \\ &= (-\Delta)\,\varphi(a). \end{aligned}$$

Thus, the escape probability is related to the solution of a Dirichlet problem.

(c) In the context of a finite connected electrical network X, if a and b are two nodes, then the *effective resistance* $r(a, b)$ between a and b is the voltage when a unit current enters a and leaves b. What is the value of $r(a, b)$?
In an electrical network, the conductance is symmetric. Define the Laplacian in terms of the symmetric conductance $t(x, y)$, that is $\Delta f(x) = \sum_{y\sim x} t(x, y)[f(y) - f(x)]$. Then this problem reduces to finding the quantity $r(a, b) = [\psi(a) - \psi(b)]$, where $\psi(x)$ is a function on X such that $(-\Delta)\,\psi(a) = 1$, $(-\Delta)\,\psi(b) = -1$, and $\Delta\psi(x) = 0$ for all x other than a and b. For this, let us consider the function φ defined as above, with $\varphi(a) = 1$, $\varphi(b) = 0$, and $\sum_{y\sim x} p(x, y)\,[\varphi(y) - \varphi(x)] = 0$ if $x \neq a, b$ where $p(x, y) = \frac{t(x,y)}{t(x)}$. For any function f on X, $\sum_x \Delta f(x) = 0$, because of the symmetry of conductance. Hence, we should have $(-\Delta)\,\varphi(a) = \alpha$ and $(-\Delta)\,\varphi(b) = -\alpha$. By the Minimum Principle, we know that $0 \leq \varphi \leq 1$ on X. Since $\varphi(a) = 1$, we calculate that $\Delta\varphi(a) = \sum_{y\sim a} t(a, y)[\varphi(y) - \varphi(a)] \leq 0$. Hence, $\alpha \geq 0$. But the assumption that $\alpha = 0$ will lead to the conclusion φ is a constant, not true. Hence, $\alpha > 0$. Consequently, $\Delta\left[\psi(x) - \frac{1}{\alpha}\varphi(x)\right] = 0$ for all $x \in X$. This means, $\psi(x) = \frac{1}{\alpha}\varphi(x) + \beta$, where β is a constant. Then, $\psi(b) = \frac{1}{\alpha}\varphi(b) + \beta = 0 + \beta$, so that $\psi(x) - \psi(b) = \frac{1}{\alpha}\varphi(x)$.
Hence, $r(a, b) = [\psi(a) - \psi(b)] = \frac{1}{\alpha}\varphi(a) = \frac{1}{\alpha} = \frac{1}{(-\Delta)\varphi(a)}$.

(d) Let X be a finite connected electrical network with symmetric conductance $t(x, y)$. X can also be considered as a random walk with transition probabilities defined by $p(x, y) = \frac{t(x,y)}{t(x)}$ for each vertex $x \in X$. Let a, b be in X. If X is considered as an electrical network, we have the notion of the effective resistance $r(a,b)$. On the other hand, if X is considered as a random walk, we have the notion of the escape probability $p_{esc}(a,b)$. Let us now give a relation between the escape probability and the effective resistance, as seen in Tetali [64] and Biggs [24].

$$\begin{aligned} p_{esc}(a,b) &= \sum_{y\sim a} p(a,y)\,[\varphi(a)-\varphi(y)] \text{ as seen in } b) \\ &= \sum_{y\sim a} \frac{t(a,y)}{t(a)}\,[\varphi(a)-\varphi(y)] \\ &= \frac{1}{t(a)} \sum_{y\sim a} t(a,y)\,[\varphi(a)-\varphi(y)] \\ &= \frac{1}{t(a)} \times \frac{1}{r(a,b)} \text{ as seen in } c) \end{aligned}$$

Thus, $t(a).p_{esc}(a,b).r(a,b)=1$.

Now, we shall get back to obtain the Dirichlet solution in a finite network X and derive some of its potential-theoretic consequences. Recall that the symmetry of the conductance in X is not assumed unless specifically stated.

Theorem 2.2.1. *(Maximum Principle for Finite Networks) Let u be defined on a finite network X. Let $A = \{y : \Delta u(y) < 0\}$. Then, $u(x) \leq \max_{y\in A} u(y)$ for all $x \in X$.*

Proof. If $A=\phi$, then $\Delta u \geq 0$ on X, so that u is subharmonic on X. If $A = X$, then u is superharmonic function on X. We have already remarked that the constants are the only superharmonic functions on a finite network (Sect. 1.2). Hence in these two cases, the theorem is trivial. Assume therefore that A is a non-empty proper subset of X.

Let $\beta = \max_{y\in A} u(y)$ and $\alpha = \max_{x\in X} u(x)$. Since $\beta \leq \alpha$, to prove the theorem, we have to show that the assumption $\beta < \alpha$ leads to a contradiction. Assume therefore that $\beta < \alpha$. Then $u(x_0) = \alpha$ for some $x_0 \in X \setminus A$. Hence, $0 \leq \Delta u(x_0) = \sum t(x_0,y)\,[u(y)-u(x_0)]$, which implies (since $u(x_0)$ is the maximum value) that $u(y) = u(x_0) = \alpha$ for all $y \sim x_0$.

Now, if $x_0 \in \partial(X\setminus A)$, then there exists some $y_1 \in A$ such that $y_1 \sim x_0$ and hence $u(y_1) = \alpha$. This shows that $\alpha \leq \beta$ contradicting the assumption that $\beta < \alpha$. Consequently, all the neighbours of x_0 are in $X \setminus A$ itself and $u(z) = \alpha$ if $z \sim x_0$. Repeat the argument starting with $u(z) = \alpha$. Then, not to contradict the assumption, we should have all the neighbours of z in $X\setminus A$ itself. Choose some $y \in A$. Since X is connected, there is a path $\{x_0, x_1, x_2, ..., x_n = y\}$ connecting x_0 and y. Let i be the largest index such that for all $j < i$, x_j is in the interior of $X \setminus A$. Note $1 \leq i \leq n-1$. Since x_{i-1} is in the interior of $X\setminus A$ and $x_i \sim x_{i-1}$, we should have $x_i \in X \setminus A$. But x_i is not in the interior of $X\setminus A$ so that $x_i \in \partial(X\setminus A)$. Hence there is some $y_1 \in A$ such that $x_i \sim y_1$. Then from what we have proved earlier $u(y_1) = \alpha$ since $u(x_i) = \alpha$. This means that $\beta \geq \alpha$ since $y_1 \in A$, a contradiction. Thus, the assumption $\alpha > \beta$ is not tenable, so that $\alpha = \beta$. That is, $\max_{x \in X} u(x) = \max_{y \in A} u(y)$. □

Corollary 2.2.2. *(Uniqueness) Let F be a proper subset of the finite network X and u be a function on X such that $\Delta u = 0$ at each vertex in F and $u = 0$ on $X \setminus F$. Then, $u \equiv 0$ on X.*

Proof. Let $A = \{x : \Delta u(x) < 0\}$ and $B = \{x : \Delta u(x) > 0\}$. Then A and B are subsets of $X \setminus F$. Now use the above theorem to conclude that if u is defined on X such that $\Delta u = 0$ on a proper subset F of X, then for any $x \in X$,

$$\min_{y \in X \setminus F} u(y) \leq \min_{y \in B} u(y) \leq u(x) \leq \max_{y \in A} u(y) \leq \max_{y \in X \setminus F} u(y).$$

Since $u = 0$ on $X \setminus F$, $u(x) = 0$ for any x in X. □

As a variation of the above Maximum Principle, we have the following Domination Principle. This is a well-known result in the classical theory of Newtonian potentials, developed in a fascinating manner by Cartan [30]. There he refers to this principle as the maximum principle. Choquet et Deny [33] call it the domination principle to avoid certain ambiguities in the terminology.

Theorem 2.2.3. *(Domination Principle) Let u and v be two functions defined on X. Let F be a non-empty subset of X. Suppose $u \geq v$ on F and $\Delta u \leq \Delta v$ on $X \setminus F$. Then, $u \geq v$ on X.*

Proof. Let $w = u - v$ on X. Then, $w \geq 0$ on F and $\Delta w \leq 0$ on $X \setminus F$. Let $A = \{y : \Delta w(y) > 0\}$. Then, $A \subset F$. Hence, by Theorem 2.2.1, $w(x) \geq \min_{y \in A} w(y)$ for every $x \in X$. That is, $w(x) \geq \min_{y \in A} w(y) \geq \min_{y \in F} w(y) \geq 0$. Hence, $u \geq v$ on X. □

For $1 \leq k \leq n-1$, let Δ_k denote a $k \times k$ submatrix of Δ, obtained by deleting $n-k$ rows and the corresponding $n-k$ columns from Δ. Then, Corollary 2.2.2 has the following equivalent formulation.

Theorem 2.2.4. *For any k, $1 \leq k \leq n-1$, Δ_k is non-singular.*

Proof. The vertices corresponding to the rows in Δ_k form a proper subset F of X. Let $u = (u_1, ..., u_k)^t$ be a column vector such that $\Delta_k u = 0$. Let $v = (v_1, ..., v_n)^t$ be a column vector of order n obtained from u by introducing 0 for the components missing in u. Now consider the column vector Δv. It can be calculated that $\Delta v(x) = \Delta_k u(x) = 0$ if $x \in F$.

Thus, v is a function defined on X such that $\Delta v = 0$ on F and $v = 0$ on $X \setminus F$. Hence by Corollary 2.2.2, $v \equiv 0$ and consequently $u \equiv 0$. Thus, we have proved that if u is a solution for the homogeneous system $\Delta_k u = 0$, then $u \equiv 0$. This implies that Δ_k is a non-singular matrix. □

The *Equilibrium Principle,* as given in Bendito et al. [18] where the Laplacian matrix is symmetric, states that if F is a proper subset of X with symmetric conductance, then there exists a unique function $\nu^F \geq 0$ on X such that $\Delta\nu^F = -1$ on F and $F = \{x : \nu^F(x) \neq 0\}$. As a generalisation, we have the following.

Theorem 2.2.5. *(Generalised Equilibrium Principle) Let F be a proper subset of X. Then, given $f \geq 0$ on F, there exists a unique $u \geq 0$ on X such that $\Delta u = -f$ on F and $u = 0$ on $X \setminus F$. Moreover, if $f > 0$, then $u > 0$ on F.*

Proof. Recall that Δ_F denotes the non-singular submatrix of Δ obtained by deleting the rows and the columns corresponding to the vertices not found in F. Then, there exists a unique function v on F such that $\Delta_F v = -f$ on F. Define u on X such that $u = v$ on F and $u = 0$ on $X \setminus F$. Then, $\Delta u = -f$ on F. Consequently, if $A = \{y : \Delta u(y) > 0\}$, then $A \subset X \setminus F$, and for any $x \in X$,

$$u(x) \geq \min_{y\in A} u(y) \text{ (Theorem 2.2.1)}$$

$$= 0 \text{ (since } u = 0 \text{ on } X \setminus F \supset A).$$

Suppose now $f > 0$ on F. Let u attain its minimum value on F at a vertex x_0. Since $u \geq 0$ on F, if $u(x_0) > 0$, then $u > 0$ on F. On the contrary, if $u(x_0) = 0$, then $\Delta u(x_0) = -f(x_0) < 0$ reads as $\sum_y t(x_0, y)[u(y) - u(x_0)] = \Delta u(x_0) = -f(x_0) < 0$, a contradiction since the left side equals $\sum_y t(x_0, y)u(y) \geq 0$.

The uniqueness of the solution follows from Corollary 2.2.2.

Theorem 2.2.6. *Let f be a function defined on X with symmetric conductance. Let $F = X \setminus \{z\}$ where z is a fixed vertex in the finite network X. Then there exists a unique function $u \in C_0$ such that $u(z) = 0$ and (when u and f are considered as column vectors) $\Delta u(x) = f(x)$ for $x \in F$, and $\Delta u(z) = -\sum_{x\in F} f(x)$.*

Proof. Write $f = f^+ - f^-$ and use Theorem 2.2.5, to see that there exists a unique function u on X such that $\Delta u(x) = f(x)$ if $x \in F$ and $u = 0$ on $X \setminus F$. That is, $\Delta u(x) = f(x)$ if $x \neq z$ and $u(z) = 0$. Now, for any function v on X, $\sum_{x\in X} \Delta v(x) = 0$. Hence, $\Delta u(z) = -\sum_{x\neq z} \Delta u(x) = -\sum_{x\neq z} f(x)$. □

At the end of Sect. 2.1, it has been mentioned that the Kirchhoff's problem in a finite electric network can be reduced to the following problem: Is it possible to determine φ on X such that $(-\Delta)\varphi = Dc$ where $c \in C_1$ and $\sum_{x\in X} Dc(x) = 0$? The answer is yes according to the following theorem.

Theorem 2.2.7. *(Solution to the Kirchhoff's Problem) Let X be a finite network with symmetric conductance. Let $g(x)$ be any function on X such that $\sum_{x\in X} g(x) = 0$. Then, there exists a function φ on X such that $\Delta\varphi(x) = g(x)$ for every x in X. This φ is unique up to an additive constant.*

Proof. Let $z \in X$. Take $f(x) = g(x)$ if $x \neq z$. Then $-\sum_{x \neq z} f(x) = g(z)$. By Theorem 2.2.6 there exists a unique $u \in C_0$ such that $u(z) = 0$ and $\Delta u(x) = g(x)$ for all $x \in X$. Consequently, if for some φ, $\Delta\varphi(x) = g(x)$ in X, then $\varphi(x) - \varphi(z) = u(x)$ for all x in X. □

Remark 2.2.1. The Kirchhoff's Problem in a finite network is a discrete analogue of the Poisson Problem of finding on a Riemannian manifold a solution u to the equation $\Delta u = f$ when f is known.

Let ω be a domain in the Euclidean space $\Re^2, n \geq 2$, or in a Riemannian manifold M. Let f be a locally Lebesgue integrable function in ω. The problems concerning the existence and the nature of a solution u satisfying the equation $\Delta u = f$ in ω (known as the Poisson equation) in the classical or in the distributions sense are important. A discrete version of the Poisson equation in an infinite tree will be discussed in Chap. 5. In a finite network X with symmetric conductance, Theorem 2.2.6 proves the existence of a solution u such that $\Delta u = f$ on X provided $\sum_{x \in X} f(x) = 0$; this solution is unique up to an additive constant. Note that a solution to the Kirchhoff's problem in a finite electric network (with symmetric conductance) can be expressed as a solution to the Poisson equation $\Delta\varphi = g$ with the restriction $\sum_{x \in X} g(x) = 0$. Theorem 2.2.7 shows that the Kirchhoff's problem has a solution.

Let F be a proper subset of X. Let f be a function on ∂F. Then the Dirichlet problem searches for a function u on F such that $u = f$ on ∂F and $\Delta u = 0$ on $\overset{0}{F}$. On the other hand, the Poisson problem searches for a solution v on X such that $\Delta v = g$ on X, where g is a function given on X; we know (Theorem 2.2.7) that a solution to the Poisson problem exists, provided $\sum_{x \in X} g(x) = 0$ in a finite network X with symmetric conductance. A solution to the combination of these two problems (namely the Dirichlet-Poisson problem) is given below by using Theorem 2.2.5; another method to solve this problem is to use the appropriate Green functions as given in Bendito et al. [18], when Δ is symmetric.

Theorem 2.2.8. *(Dirichlet-Poisson equation) Let F be a proper subset of a finite network X. Let f and g be two functions on X. Then, there exists a unique function u on X such that $\Delta u = f$ on F and $u = g$ on $X \setminus F$.*

Proof. Let $h = \Delta g$ on X. Let us consider the functions h^+ and h^- and use Theorem 2.2.5 and the fact that Δ is linear to construct a function ψ_0 on X such that $\Delta\psi_0 = h$ on F and $\psi_0 = 0$ on $X \setminus F$. Similarly, let us construct φ_0 on X such that $\Delta\varphi_0 = f$ on F and $\varphi_0 = 0$ on $X \setminus F$. Write $u = \varphi_0 + g - \psi_0$ on X. Then, $\Delta u = f + h - h = f$ on F and $u = 0 + g - 0 = g$ on $X \setminus F$. Thus, the existence of a solution u is proved. For the uniqueness of this solution, we use the minimum principle (Corollary 2.2.2). □

Corollary 2.2.9. *(Classical Dirichlet Problem) Let g be defined on the boundary ∂F of a proper finite subset F. Then, there exists a unique function u on F such that $\Delta u(x) = 0$ at every $x \in \overset{0}{F}$ and $u = g$ on ∂F.*

Proof. Extend g on X by giving arbitrary values outside ∂F. Take $f \equiv 0$ and apply the above theorem with respect to $\overset{\circ}{F}$ to assert the existence of the solution u. The uniqueness of u follows from the Maximum Principle. □

Remark 2.2.2. 1. To determine the unique solution u in the above theorem, we proceed as follows: Let the number of vertices in F be k. Since $u = g$ on $X \setminus F$, we have only to find the k values of u on F. Denoting them by $(u_1, u_2, ..., u_k)$, if we take out the k linear equations in $\Delta u = f$ on F, then we have a linear system given by $\Delta_F(u_1, u_2, ..., u_k)^t = (\alpha_1, \alpha_2, ..., \alpha_k)^t$ where the right side is known. We also know that Δ_F is invertible. Hence, $u_1, u_2, ..., u_k$ can be calculated.

2. The proof of the above Corollary 2.2.9 is given by using potential-theoretic methods. An alternate method of solving the Dirichlet problem for a special type of finite subsets of a tree T is given by Berenstein et al. [21, p.461] by using the hitting distribution of the stochastic process generated by the transition probability structure of T.

The above theorem asserting the unique solution of the Dirichlet-Poisson equation is very fundamental in proving certain basic principles and results (in electrical networks and random walks) such as the condenser principle, the equilibrium principle, balayage, the Green's function, the Poisson kernel etc., as well as the earlier-mentioned concepts like effective resistance, escape probability, hitting time etcetera.

Theorem 2.2.10. *(Condenser Principle) Let A and B be two non-empty disjoint subsets of X. Let $F = X \setminus (A \cup B) \neq \phi$. Let a and b be two real numbers, $a < b$. Then, there exists a unique φ on X such that $a \leq \varphi(x) \leq b$ if $x \in X$,*

$$\varphi(x) = a \text{ and } \Delta\varphi(x) \geq 0 \text{ if } x \in A,$$

$$\varphi(x) = b \text{ and } \Delta\varphi(x) \leq 0 \text{ if } x \in B$$

$$\text{and } \Delta\varphi(x) = 0 \text{ if } x \in F.$$

Proof. In Theorem 2.2.8, take two functions f and g on X such that $f = 0$ on F and $g(x) = a$ if $x \in A$ and $g(x) = b$ if $x \in B$. Then there exists a unique function φ on X such that $\varphi(x) = a$ if $x \in A$, $\varphi(x) = b$ if $x \in B$, and $\Delta\varphi(x) = 0$ if $x \in F$. Then, $a \leq \varphi(x) \leq b$ if $x \in X$ (see the proof of Corollary 2.2.2). Consequently, the inequalities $\Delta\varphi \geq 0$ on A and $\Delta\varphi \leq 0$ on B follow from the definition of Δ. □

Let us consider now some potential-theoretic problems with Dirichlet solutions associated with the operator Δ. An important problem in an electrical network is: given a potential function u on X and a subset F of X, is it possible to sweep out the charges associated with u onto F so that the potential function v associated with the new distribution of charges preserves the same values on F?

Theorem 2.2.11. *(Balayage) Let F be a proper subset of X and let u be a function on X such that $\Delta u \leq 0$ on $X \setminus F$. Then there exists a unique function v on X such that $v \leq u$ on X, $v = u$ on F, and $\Delta v = 0$ on $X \setminus F$ so that* $\sum_{y \in X \setminus \overset{0}{F}} \Delta u(y) = \sum_{s \in \partial F} \Delta v(s)$.

Proof. By using Theorem 2.2.8, we construct a function φ on X such that $\Delta\varphi = \Delta u$ on $X \setminus F$ and $\varphi = 0$ on F. Let $v = u - \varphi$ on X. Then, $v = u$ on F and $\Delta v = \Delta u - \Delta\varphi = 0$ on $X \setminus F$.

Let $w = u - v$ on X. Then, $\Delta w = \Delta u - \Delta v \leq 0$ on $X \setminus F$ by hypothesis, and $w = 0$ on F. Hence, by Theorem 2.2.1, $w \geq 0$ on X, that is $u \geq v$ on X. Further, since $\sum_{x \in X} \Delta u(x) = \sum_{x \in X} \Delta v(x) = 0$ and since $\sum_{x \in \overset{0}{F}} \Delta u(x) = \sum_{x \in \overset{0}{F}} \Delta v(x)$, we have

$$\begin{aligned}
\sum_{x \in X \setminus \overset{0}{F}} \Delta u(x) &= \sum_{x \in X \setminus \overset{0}{F}} \Delta v(x) \\
&= \sum_{x \in \partial F} \Delta v(x) + \sum_{x \in X \setminus F} \Delta v(x) \\
&= \sum_{x \in \partial F} \Delta v(x) + 0.
\end{aligned}$$

□

Remark 2.2.3. In the context of finite electrical networks, if we interpret $-\Delta u(x)$ as the charge associated with u at the vertex x, then the last assertion indicates that the total charge of u outside F has been swept (*balayée*) onto ∂F to obtain v.

In the study of electrical networks, the existence of a non-negative function with an associated point charge is very helpful. Such a function cannot be defined on X, if X is a finite network. However, on every proper subset of X, this function can be defined.

Theorem 2.2.12. *(Green's function) Let F be a proper subset of X. If $y \in \overset{0}{F}$, then there exists a unique function $G_y^F(x) \geq 0$ on X such that $\Delta G_y^F(x) = -\delta_y(x)$ if $x \in \overset{0}{F}$, $G_y^F(s) = 0$ if $s \in X \setminus \overset{0}{F}$, and $G_y^F(x) \leq G_y^F(y)$ for any $x \in X$.*

Proof. By Theorem 2.2.8, there exists a unique function on X, noted $G_y^F(x)$ such that $\Delta G_y^F(x) = -\delta_y(x)$ if $x \in \overset{0}{F}$ and $G_y^F(s) = 0$ if $s \in X \setminus \overset{0}{F}$. Moreover, since $\Delta G_y^F(x) \leq 0$ on $\overset{0}{F}$ and $G_y^F(x) = 0$ if $x \in X \setminus \overset{0}{F}$, we conclude that $G_y^F \geq 0$ on X. (It can be seen that $G_y^F > 0$ on $\overset{0}{F}$, if $\overset{0}{F}$ is connected.)

To prove the last assertion, note that $\Delta G_y^F(x) \geq 0$ if $x \neq y$. Hence in Theorem 2.2.3 (Domination Principle), if we take $v(x) = G_y^F(x)$ and $u(x) = G_y^F(y)$, a constant, then $u(x) = v(x)$ if $x = y$ and $\Delta u(x) = 0 \leq \Delta v(x)$ if $x \neq y$. Consequently, $u \geq v$ on X, that is $G_y^F(y) \geq G_y^F(x)$ on X. □

Theorem 2.2.13. *(Poisson Kernel) Let X be a finite network with symmetric conductance. Let F be a proper subset of X. Let $G_x^F(y)$ be the Green function on F for $x \in \overset{0}{F}$ and $y \in F$. Let $f(s)$ be defined on ∂F. Then the Dirichlet solution in F with boundary value $f(s)$ is $\sum_{s \in \partial F} f(s) \frac{\partial G_x^F}{\partial n^-}(s)$.*

Proof. Start with the Green's identity (Corollary 1.3.3) for any two real-valued functions u and v defined on F,

$$\sum_{y \in \overset{0}{F}} [u(y)\Delta v(y) - v(y)\Delta u(y)] = -\sum_{s \in \partial F} [u(s)\frac{\partial v}{\partial n^-}(s) - v(s)\frac{\partial u}{\partial n^-}(s)].$$

Let now $h(x)$ be the Dirichlet solution in F with boundary value $f(s)$. Take $v = h$ and $u = G_x^F$ in the above equation. Since $\Delta h(y) = 0$ if $y \in \overset{0}{F}$, $h(s) = f(s)$ if $s \in \partial F$, $G_x^F(s) = 0$ if $x \in \overset{0}{F}$, $s \in \partial F$, and $\Delta G_x^F = -\delta_x$ if $x \in \overset{0}{F}$, we obtain $h(x) = \sum_{s \in \partial F} f(s) \frac{\partial G_x^F}{\partial n^-}(s)$. □

Definition 2.2.1. Let X be a finite network with symmetric conductance and let F be a proper subset of X. For $x \in \overset{0}{F}$ and $s \in \partial F$, the Poisson kernel for F is defined as $P(x,s) = \frac{\partial G_x^F}{\partial n^-}(s)$.

Remark 2.2.4. Alternatively, for $x \in \overset{0}{F}$ and $s \in \partial F$, the Poisson kernel $P(x,s)$ for fixed $s \in \partial F$ can be defined as the Dirichlet solution on F with boundary value $\delta_s(z)$ for $z \in \partial F$. This way, we can avoid introducing the Green's identity to define the Poisson kernel. Consequently, if F is a proper subset of a finite network X with or without symmetric conductance, and if $f(s)$ is a real-valued function on ∂F, then $h(x) = \sum_{s \in \partial F} P(x,s) f(s)$ is the Dirichlet solution in F with boundary value $f(s)$.

Proposition 2.2.14. *Let F be a proper subset of a finite network X with symmetric conductance. If $a, b \in \overset{0}{F}$, then $G_a^F(b) = G_b^F(a)$.*

Proof. This follows immediately from the Green's identity involving u and v, if we set $u(x) = G_a^F(x)$ and $v(x) = G_b^F(x)$. □

Theorem 2.2.15. *(Mixed boundary-value problem) Let F be a proper subset of a finite network X. Let A and B be non-empty disjoint subsets such that $A \bigcup B = \partial F$. Let f be a function defined on F. Then there exists a unique function u on F such that $\Delta u(x) = f(x)$ if $x \in \overset{0}{F}$, $u(s) = f(s)$ if $s \in A$ and $\frac{\partial u}{\partial n^-}(s) = f(s)$ if $s \in B$.*

Proof. Let $\Delta^{\bullet}_F$ be the Laplacian restricted to F. Assume that f is arbitrarily extended outside F to cover $X \setminus F$. Then, there exists (as in Theorem 2.2.8) a unique function u on F such that $u = f$ on A and $\Delta^{\bullet}_F u = f$ on $F \setminus A$. Since $\Delta u(x) = \Delta^{\bullet}_F u(x)$ if $x \in \overset{0}{F}$ and $\Delta^{\bullet}_F u(s) = \frac{\partial u}{\partial n^-}(s)$ if $s \in \partial F$, we have the following properties for u, namely $\Delta u(x) = f(x)$ if $x \in \overset{0}{F}$, $u(s) = f(s)$ if $s \in A$ and $\frac{\partial u}{\partial n^-}(s) = f(s)$ if $s \in B$. □

Theorem 2.2.16. *(Neumann problem) Let F be a proper subset of a finite network X with symmetric conductance and let $f(s)$ be a function defined on ∂F. Then, the following two statements are equivalent:*

i. There exists a function u on F such that $\Delta u(x) = 0$ if $x \in \overset{0}{F}$ and $\frac{\partial u}{\partial n^-}(s) = f(s)$ if $s \in \partial F$.

ii. $\sum_{s \in \partial F} f(s) = 0$.

Proof. Suppose u exists with the stated properties. Then, by Proposition 1.3.1, $\sum_{s \in \partial F} f(s) = \sum_{s \in \partial F} \frac{\partial u}{\partial n^-}(s) = -\sum_{x \in \overset{o}{F}} \Delta u(x) = 0$. Conversely, if $\sum_{s \in \partial F} f(s) = 0$, then extend f on ∂F by giving value 0 at each vertex in $\overset{0}{F}$. Then, by Theorem 2.2.7, there exists a function u on F (unique up to an additive constant) such that $\Delta^{\bullet}_F u = f$ on F. This function u has the following properties: $\Delta u(x) = 0$ if $x \in \overset{0}{F}$ and $\frac{\partial u}{\partial n^-}(s) = f(s)$ if $s \in \partial F$. □

2.3 Dirichlet Semi-Norm

In this section, X is assumed to be a finite network with symmetric conductance. An interesting alternative method to carry out the potential-theoretic study of functions on a network with symmetric conductance is to consider this set of functions as an inner-product space and obtain the basic Dirichlet solution as a projection. Then, we can prove other results as consequences of the existence of solutions to appropriate Dirichlet problems. For a pair of functions u and v on a subset E of a finite network X with symmetric conductance, define an inner-product (Sect. 1.3)

$$(u, v)_E = \frac{1}{2} \sum_{x,y \in E} t(x, y)[u(x) - u(y)][v(x) - v(y)].$$

Write $\|u\|_E^2 = (u, u)_E$. Let us suppose that E is connected. Since $\|u\|_E = 0$ implies that u is a constant on E, $\|u\|_E$ is a semi-norm which is called the *Dirichlet semi-norm* on E. Let $\mathcal{F}$ denote the equivalence classes $f^{\sim}$ of functions on E, so that two functions on E are in the same class if and only if they differ by a constant. Note that $\mathcal{F}$ is an inner-product space; if u and v are any two functions defined on E and if $u^{\sim}$ and $v^{\sim}$ are the equivalence classes defined by u and v, then $\|u^{\sim}\| = \|u\|$

and $\|u^{\sim} - v^{\sim}\| = \|u - v\|$. Let $F \subset \overset{0}{E}$. Let H denote the subspace of $\mathcal{F}$ such that each $h \in H$ is harmonic at every vertex of F, that is $\Delta h(x) = 0$ if $x \in F$.

Theorem 2.3.1. *H is a closed subspace of $\mathcal{F}$.*

Proof. Let $h_n^{\sim} \in H$ be a Cauchy sequence in $\mathcal{F}$. For each equivalence class $h_n^{\sim}$, extract a function h_n from the class $h_n^{\sim}$ so that $h_n(e) = 0$ where e is a fixed vertex in E. Since $\|h_n - h_m\| = \|h_n^{\sim} - h_m^{\sim}\| \to 0$ when $n, m \to \infty$, for any $x \in E$,

$$\frac{1}{2}t(x,e)[(h_n - h_m)(e) - (h_n - h_m)(x)]^2 \le \|h_n - h_m\|^2 \to 0$$

when $n, m \to \infty$. Since $(h_n - h_m)(e) = 0$, we deduce that the sequence $\{h_n(x)\}$ converges at every $x \sim e$. Then, we show that there is convergence at all the neighbours of each $x \sim e$. Thus proceeding, we see that h_n converges on E. If we write $h(x) = \lim_n h_n(x)$, then $h(x)$ is harmonic at each vertex in F, and $\|h_n - h\| \to 0$. Hence $h_n^{\sim} \to h^{\sim}$; that is, H is closed in $\mathcal{F}$. □

Using the properties of inner-product spaces, we conclude from the above theorem: For every $f^{\sim} \in \mathcal{F}$, there exists a unique $h^{\sim} \in H$ such that $\|h^{\sim} - f^{\sim}\|$ is minimum; $h^{\sim}$ is the projection of $f^{\sim}$ on H, so that $\|h^{\sim}\| \le \|f^{\sim}\|$ and $\|h^{\sim}\| = \|f^{\sim}\|$ if and only if $f^{\sim} \in H$. Let us denote by $\mathcal{F}^*$ the subspace of $\mathcal{F}$ such that $f^{\sim} \in \mathcal{F}^*$ if and only if the equivalence class represented by $f^{\sim}$ contains a function f on E which is 0 on $E \setminus F$.

Theorem 2.3.2. *$\mathcal{F}^*$ is the orthogonal complement of H in $\mathcal{F}$; that is $\mathcal{F} = \mathcal{F}^* \oplus H$, and $\mathcal{F}^* \perp H$.*

Proof. Remark that if $f_1^{\sim}$ and $f_2^{\sim}$ are in $\mathcal{F}$, and if we take some f_1 (respectively f_2) from the equivalence class $f_1^{\sim}$ (respectively $f_2^{\sim}$), then $(f_1, f_2)_E$ is independent of the choice of the functions f_1 and f_2. We define $(f_1^{\sim}, f_2^{\sim})_E = (f_1, f_2)_E$. Now H is orthogonal to $\mathcal{F}^*$. For, let $f^{\sim} \in \mathcal{F}^*$ and $h^{\sim} \in H$. Choose f from the class $f^{\sim}$ such that $f = 0$ on $E \setminus F$. Choose some h from the class $h^{\sim}$. Note that in Theorem 1.3.2, we have proved $\sum_{x \in \overset{0}{E}} f(x)\Delta g(x) + (f,g)_E = -\sum_{s \in \partial E} f(s)\frac{\partial g}{\partial n^-}(s)$; in this, take $g = h$. Now $\Delta h(x) = 0$ if $x \in F$and $f = 0$ on $E \setminus F \supset (\overset{0}{E} \setminus F) \bigcup \partial E$. Consequently, $(f,h)_E = (f,g)_E = 0$. Hence, $(f^{\sim}, g^{\sim})_E = 0$.

Suppose $f^{\sim} \in \mathrm{H} \cap \mathcal{F}^*$. If f is a function such that $\Delta f = 0$ on F and $f = 0$ on $E \setminus F$, then $f \equiv 0$ (Corollary 2.2.2), so that $f^{\sim} = 0^{\sim}$. Let now $f^{\sim} \in \mathcal{F}$. Choose some f in the class $f^{\sim}$. Let h be the function such that $\Delta h = 0$ on F and $h = f$ on $E \setminus F$ (Theorem 2.2.8). Then, $[f - h]^{\sim} \in \mathcal{F}^*$, $h^{\sim} \in H$ and $f^{\sim} = [f - h]^{\sim} + h^{\sim}$. □

The following is another version of the above result, without any reference to the equivalence classes, to obtain the Dirichlet Principle in the context of a finite network with symmetric conductance (see also, [55]).

Theorem 2.3.3. *(Dirichlet Principle) Let E be a connected set and $F \subset \overset{0}{E}$. Let f be a function defined on $E \setminus F$. Let Γ be the family of functions g on E such that $g = f$ on $E \setminus F$. Then, there exists a function h in Γ such that for any $g \in \Gamma$, $\|h\|_E \leq \|g\|_E$ and $\|h\|_E = \|g\|_E$ if and only if $h \equiv g$. This function h is uniquely determined and $\Delta h = 0$ at every vertex of F.*

Proof. Let h be the unique function on E (Theorem 2.2.8), such that $h = f$ on $E \setminus F$ and $\Delta h(x) = 0$ if $x \in F$. Then, we prove as above, by using the Green's formula, that if $g \in \Gamma$, then $(g-h) \perp h$. Hence, writing $g = (g-h)+h$, we conclude $\|g^{\sim}\|_E \geq \|h^{\sim}\|_E$ which implies that $\|g\|_E \geq \|h\|_E$. Suppose $\|g\|_E = \|h\|_E$; then, $\|g^{\sim}\|_E = \|h^{\sim}\|_E$ which implies that $g^{\sim} = h^{\sim}$ by the property of projection. This means that $g - h$ is a constant on E. Since $g - h = 0$ on $E \setminus F$, $g \equiv h$. □

Remark 2.3.1. The above Dirichlet Principle is well-known in the classical case. Let Ω be a bounded domain in the Euclidean space $\Re^n, n \geq 2$. In the space of continuous functions on Ω which admit finite continuous (Lebesgue) square-summable gradient, define the scalar product $(f, g) = \int_\Omega (gradf, gradg)dx$, with the corresponding Dirichlet semi-norm $\|f\|$. Suppose a continuous function f is defined on $\partial\Omega$. Let us extend f as a continuous function g on Ω for which $\|g\|$ is defined. Let us place all such extended functions g in the class Γ. If there is some $h \in \Gamma$ with minimum norm, it will prove to be the Dirichlet solution in Ω with boundary value f on $\partial\Omega$. But, it is possible that there is no such extension $h \in \Gamma$ on Ω for which $\|h\| = \inf_{\Gamma} \|g\|$. That is, the classical Dirichlet problem is not solvable in an arbitrary bounded domain in $\Re^n$. As an example, cite the domain $0 < |x| < 1$ in $\Re^2$.

However, in a network, the solution to the classical Dirichlet problem can always be obtained as a projection. For, let E be a finite set of connected vertices in a network X with symmetric conductance. Let f be a function defined on ∂E. Giving arbitrary values at the vertices in $\overset{0}{E}$, we can assume that f is defined on E. Then (by Theorem 2.3.2, taking $F = \overset{0}{E}$) $f^{\sim}$ can be written uniquely as $f^{\sim} = h^{\sim} + g^{\sim} \in H \oplus \digamma^*$. Take a function u in the equivalence class $h^{\sim}$. Then, by the definition of $\digamma^*$, $f - u$ is a function on E taking a constant value c on ∂E. Consequently, $h = u + c$ is a function defined on E such that $\Delta h = 0$ at every vertex of $\overset{0}{E}$ and $h = f$ on ∂E.

2.4 Schrödinger Operators on Finite Networks

In this section, we study the Schrödinger operators on a finite network X on the lines of the earlier study with respect to the Laplace operator. The Laplacian is Δ and the Schrödinger operator is the q-Laplacian Δ_q given by $\Delta_q u(x) = \Delta u(x) - q(x)u(x)$, where $q(x) \geq 0$ is defined on the vertices of the network; when we write Δ_q, it is assumed that $q(x) > 0$ for at least one vertex. (In Chap. 4, in the context of infinite networks, we relax the condition $q \geq 0$ and allow q to take negative values in a restricted range.) Note

$$\begin{aligned} \Delta_q u(x) &= \Delta u(x) - q(x)u(x) \\ &= \sum_y t(x,y)[u(y) - u(x)] - q(x)u(x) \\ &= \sum_y t(x,y)u(y) - [t(x) + q(x)]u(x), \end{aligned}$$

$$\text{where } t(x) = \sum_y t(x,y) \text{ for every } x \text{ in } X.$$

We identify Δ_q as an $n \times n$ matrix where n is the number of vertices in X. Let Δ_q^l be a square matrix of order l, $1 \leq l \leq n$, formed by l rows and the corresponding l columns of the matrix Δ_q. By using a minimum principle on X, we prove that Δ_q^l is a non-singular matrix which fact is fundamental in solving a generalised version of the classical Dirichlet problem in a network. Then, by characterizing Green's function, Poisson kernel, condenser principle and balayage as Dirichlet problems with suitable boundary conditions, we are able to deduce immediately these potential-theoretic results in a finite network. Thus, these results are obtained effectively by calculating the inverses of suitable submatrices Δ_q^l.

Bendito et al. [20] prove many of these results in a finite network X with symmetric conductance, by computing appropriate equilibrium measures: the equilibrium measure for a proper subset F of X is a function $\mu \geq 0$ on X such that $\Delta_q \mu(x) = -1$ for $x \in F$ and $\{x : \mu(x) \neq 0\} \subset F$. (It is proved that μ is unique and $\{x : \mu(x) \neq 0\} = F$.) The effective computation of such measures is accomplished by solving linear programming problems. In another direction, interpreting the above problems in the context of probability theory (random walks), solutions are obtained by Chung and Yau [35], Doyle and Snell [44], Tetali [64] and others.

The theory concerning the Schrödinger operator Δ_q, where $q(x) \geq 0$ for all x and $q(x_0) > 0$ for at least one x_0 in X, can be developed on the same lines as for the Laplace operator Δ by using matrices. However, we use a different approach here based on the properties of the Δ_q-superharmonic functions on X. Such a method is not available for the Laplace operator, since any Δ-superharmonic function on a finite network X is constant. Actually, by introducing the Δ_q-superharmonic functions at this stage, we try to bring into focus the potential-theoretic methods that will be used later in the context of infinite networks.

Lemma 2.4.1. *Suppose $\Delta_q u \leq 0$ on X. Then $u \geq 0$ on X.*

Proof. For, suppose u takes negative values on X, let u attain its minimum value $-m$ at z. Since $0 \geq \Delta_q u(z) = \sum_{y \sim z} t(z,y)[u(y) - u(z)] - q(z)u(z)$, if we assume $q(z) > 0$, then we find $\Delta_q u(z) > 0$, a contradiction. Hence $q(z) = 0$ and as a consequence $u(y) = u(z)$ for $y \sim z$. Since q does not vanish everywhere, there is some vertex a where $q(a) > 0$. Let $\{z = x_0, x_1, ..., x_j = a\}$ be a path joining z and a. Let i be the smallest index such that $q(x_i) = 0$ and $q(x_{i+1}) > 0$. Then, by the above argument we should have $u(x_{i+1}) = -m$ since $x_{i+1} \sim x_i$. Consequently,

$$0 \geq \Delta_q u(x_{i+1}) = \sum_{y \sim x_{i+1}} t(x_{i+1}, y)[u(y) - u(x_{i+1})] - q(x_{i+1})u(x_{i+1})$$
$$= \text{[a non-negative quantity]-[a positive quantity]} \times \text{[a negative quantity]} > 0,$$

a contradiction. Hence, $u \geq 0$ on X. □

Remark 2.4.1. The above lemma implies that if $\Delta_q u = 0$ on X, then $u \equiv 0$; however this conclusion is not valid if $q \equiv 0$, but what is true in this case is that u will be a constant.

Theorem 2.4.2. *The matrix Δ_q is non-singular.*

Proof. If the vertices of X are denoted by $x_1, x_2, ..., x_n$, and if $u(x)$ is a function defined on X, then let us write $u = (u_1, u_2, ..., u_n)^t$ as a column vector where $u_i = u(x_i)$. Now, by the above Lemma 2.4.1, the equation $\Delta_q u = 0$ has the unique solution $u \equiv 0$. Hence, Δ_q is a non-singular matrix. □

Remark 2.4.2. To prove the above Theorem 2.4.2, we have used the fact that the non-negative function $q(x)$ on X is not identically 0. However, we shall see in Chap. 4 in the context of infinite networks that q can be allowed to take some negative values in a restricted range as remarked by Bendito et al. [20] in the case of finite networks with symmetric conductance. In fact, they assume that there exists a function $\xi(x) > 0$ on the finite network X such that $q(x) \geq \frac{\Delta\xi(x)}{\xi(x)}$ for every x in X. This means that $q(x)$ can possibly take negative values.

Lemma 2.4.3. *(q-Minimum principle for Schrödinger operators) Let F be a proper subset of X. Suppose u is a function on X such that $\Delta_q u \leq 0$ at every vertex in F and $u \geq 0$ on $X \setminus F$. Then $u \geq 0$ on X. (In particular, if $\Delta_q v = 0$ at every vertex in F and $v = 0$ on $X \setminus F$, then $v \equiv 0$.)*

Proof. Suppose u takes a negative value on X. If $-m = \min_{x \in X} u(x)$, then $u(x_0) = -m$ for some $x_0 \in F$. Since F is a proper subset of X, there is some $a \in X \setminus F$. Let $\{x_0, x_1, ..., x_j = a\}$ be a path connecting x_0 and a. Let i be the smallest index such that $x_i \in F$ and $x_{i+1} \in X \setminus F$. Then, an argument similar to the one in the proof of Lemma 2.4.1 shows that the assumption that u takes on negative values is not tenable. We conclude $u \geq 0$ on X. □

Theorem 2.4.4. *For any l, $1 \leq l \leq n$, the matrix Δ_q^l is non-singular.*

Proof. Since we have already proved the theorem when $l = n$ (Theorem 2.4.2), we shall suppose $l < n$. Let F be the subset $x_1^{'}, x_2^{'}, ..., x_l^{'}$ corresponding to the l rows of Δ_q^l. Consider the equation $\Delta_q^l u^{'} = 0$, where $u^{'} = (u_1^{'}, ..., u_l^{'})^t$ is arbitrary.

Let $u = (u_1, u_2, ...u_n)^t$, obtained from $u^{'}$ by introducing $(n - l)$ extra zeros at the appropriate places. Note now $\Delta_q u = 0$ on F and $u = 0$ on $X \setminus F$. Hence, by the q-Minimum Principle (Lemma 2.4.3), $u \equiv 0$ and hence $u^{'} \equiv 0$. This proves that the matrix Δ_q^l is non-singular. □

Theorem 2.4.5. *(q-Domination Principle for Schrödinger operators) Let v and f be two functions on X. Let $\Delta_q v \leq 0$ on X and $A = \{x : \Delta_q f(x) < 0\}$. If $v \geq f$ on A, then $v \geq f$ on X.*

Proof. Let $u = v - f$. Write $F = X \setminus A$. If $F = \phi$, then there is nothing to prove. If $F = X$, then $\Delta_q f(x) \geq 0$ for all x. Hence $\Delta_q u(x) \leq 0$ for all $x \in X$. Then, $u \geq 0$ on X (Lemma 2.4.1). If F is a proper subset of X, then on $F = X \setminus A$, $\Delta_q u \leq 0$ and on A, $u \geq 0$. Hence, by the above Lemma 2.4.3, $u \geq 0$ on X. □

Theorem 2.4.6. *(Dirichlet-Poisson solution for Schrödinger operators) Let F be a subset of X. Given two functions f on F and g on $X \setminus F$, there exists a unique function u on X such that $\Delta_q u = -f$ on F and $u = g$ on $X \setminus F$. Moreover, if f and g are non-negative, then $u \geq 0$ on F.*

Proof. If $F = X$, then Theorem 2.4.2 establishes the existence of the unique solution u and then Lemma 2.4.1 proves that $u \geq 0$ on X, if $f \geq 0$.

Let us suppose that F is a proper subset of X. Let $u = (u_1, u_2, ..., u_n)^t$ and $v = (v_1, v_2, ..., v_n)^t$ be the column vectors such that $u(x) = g(x)$ if $x \in X \setminus F$ and $v(x) = -f(x)$ if $x \in F$. Now, if we write $\Delta_q u = v$, then (Δ_q being a non-singular matrix) we can first determine the value of $u(x)$ for $x \in F$. Consequently, we have all the values which u takes on X such that $u = g$ on $X \setminus F$ and $\Delta_q u = -f$ on F. If f and g are non-negative, then $\Delta_q u \leq 0$ on F and $u \geq 0$ on $X \setminus F$. Hence by Lemma 2.4.3, $u \geq 0$ on X.

To prove the uniqueness of u, suppose s is another function such that $s = g$ on $X \setminus F$ and $\Delta_q s = -f$ on F. Then $\varphi = s - u$ satisfies the conditions $\varphi = 0$ on $X \setminus F$ and $\Delta_q \varphi = 0$ on F. Then by Lemma 2.4.1, $\varphi \equiv 0$. □

Corollary 2.4.7. *(q-Poisson solution) There is always a unique solution to the Poisson equation, namely, given f on X, there exists a unique u on X such that $\Delta_q u = f$ on X.*

Proof. This is a consequence of the fact that the matrix Δ_q is non-singular. □

Let us solve now the Dirichlet problem for the Schrödinger operator. We can use as for the Laplace operator (Remark 1 following Corollary 2.2.9), the non-singularity property of the matrix Δ_q and any of its principal submatrices to obtain the solution. However, we adapt a method from the classical potential theory in $\Re^n$ to solve this problem in a finite network X. The advantage in this method is that it works even if the network X is not a finite set, as we shall see in the next chapter.

A function u on a subset F of X is said to be q-*superharmonic* (respectively, q-harmonic, q-subharmonic) on F if and only if $\Delta_q u \leq 0$ (respectively, $\Delta_q u = 0$,

$\Delta_q u \geq 0$) at every vertex of $\overset{0}{F}$. However, we use the expression that u is q-harmonic at a vertex a, if u is defined on $V(a)$ and $\Delta_q u(a) = 0$. (For the Laplace operator on a finite network on X, $\Delta u \leq 0$ on X if and only if u is a constant on X. Consequently, the use of classical potential-theoretic methods, involving superharmonic functions defined by the Laplacian, to study the functions on a finite network is limited. In that case, a recourse to matrices is convenient, as we have seen earlier.)

1. If u and v are q-superharmonic on F, then $s = \inf(u, v)$ is q-superharmonic on F.
   For, if $x \in \overset{0}{F}$, then

$$\begin{aligned}\Delta_q s(x) &= \sum_y t(x,y)s(y) - [t(x) + q(x)]\, s(x) \\ &= \sum_y t(x,y)s(y) - [t(x) + q(x)]\, v(x), \text{assuming } s(x) = v(x) \\ &\leq \sum_y t(x,y)v(y) - [t(x) + q(x)]\, v(x) = \Delta_q v(x) \leq 0.\end{aligned}$$

2. If u_n is a sequence of q-superharmonic (respectively, q-harmonic) functions on F, and if $u(x) = \lim_{n\to\infty} u_n(x)$ exists and is finite for every $x \in F$, then u is q-superharmonic (respectively, q-harmonic) on F.
   For, if $x \in \overset{0}{F}$, then $\sum_y t(x,y)u_n(y) - [t(x) + q(x)]\, u_n(x) \leq 0$.
   Taking limits when $n \to \infty$, we obtain

$$\sum_y t(x,y)u(y) - [t(x) + q(x)]\, u(x) \leq 0.$$

   That is, u is q-superharmonic on F (Similar proof in the case of a sequence of q-harmonic functions.)

Δ_q-Poisson modification. For a function u on a subset E of a network X, and an arbitrary vertex $a \in \overset{0}{E}$, let us write $P_a u(x) = u(x)$ if $x \neq a$ and $P_a u(a) = \sum_z \frac{t(a,z)}{t(a)+q(a)} u(z)$. Let us call $P_a u$ the Δ_q-Poisson modification of u at a. Note that $P_a u(x)$ is q-harmonic at $x = a$.

Lemma 2.4.8. *Assume that u is q-superharmonic on E. Let $a \in \overset{0}{E}$. Then $P_a u$ is q-superharmonic on E, $P_a u(x)$ is q-harmonic at $x = a$, and $P_a u \leq u$ on E.*

Proof. Since u is q-superharmonic at a , we have $P_a u(a) \leq u(a)$. For $x \notin V(a)$, we have $P_a u = u$ on $E \cap V(x)$, so that $\Delta_q P_a u(x) = \Delta_q u(x) \leq 0$ at each $x \in \overset{0}{E} \setminus V(a)$.

Let us take the case when $x \in \overset{0}{E} \cap V(a)$. Then there are two cases to consider, namely $x = a$ or $x \neq a$.
For $x \neq a$, we have

$$\begin{aligned}\Delta_q P_a u(x) &= -[t(x)+q(x)]P_a u(x) + \sum_z t(x,z)P_a u(z)\\ &\leq -[t(x)+q(x)]u(x) + \sum_z t(x,z)u(z)\\ &= \Delta_q u(x) \leq 0.\end{aligned}$$

For $x = a$, we have

$$\begin{aligned}\Delta_q P_a u(a) &= -[t(a)+q(a)]P_a u(a) + \sum_z t(a,z)P_a u(z)\\ &= -\sum_z t(a,z)u(z) + \sum_z t(a,z)u(z) = 0.\end{aligned}$$

Thus, for all $x \in \overset{0}{E}$, we have $\Delta_q P_a(x) \leq 0$. Hence $P_a u$ is q-superharmonic on E. □

Δ_q-Perron family. Let F be a subset of X and E be some subset contained in $\overset{0}{F}$. A non-empty family $\mathcal{F}$ of functions on F is said to be a (lower directed) *Δ_q-Perron family* on (F, E) if it satisfies the following conditions.

1. For any $v_1, v_2 \in \mathcal{F}$, there exists $v \in \mathcal{F}$ such that $v \leq \min(v_1, v_2)$.
2. $P_a v \in \mathcal{F}$, for every $v \in \mathcal{F}$ and $a \in E$; also $P_a v \leq v$.
3. There exists a function u_0 on F such that $v \geq u_0$ for all $\mathcal{F}$.

Remark 2.4.3. 1. Analogously, we define an upper directed Δ_q-Perron family of functions on (F, E).
2. We define also, with respect to the Laplacian operator Δ, lower directed and upper directed Δ-Perron families on (F, E), by taking $q \equiv 0$.

Example of a Δ_q-Perron family: Let u be a q-subharmonic function on a set F. Let $\mathcal{F}$ be the family of q-superharmonic functions v on F such that $v \geq u$. Suppose this family of functions is non-empty. Then, $\mathcal{F}$ is a lower-directed family of q-superharmonic functions on $(F, \overset{0}{F})$.

Theorem 2.4.9. *If $\mathcal{F}$ is a lower directed Δ_q-Perron family on (F, E), then $h(x) = \inf v(x)$, $x \in F$, $v \in \mathcal{F}$, is q-harmonic at every vertex of E.*

Proof. Since each v in $\mathcal{F}$ majorizes u_0, we have $h(x) \geq u_0(x)$ in $V(E)$. Let $a \in E$ be fixed arbitrarily. Since $V(a)$ is a finite set, we can find a sequence $\left\{v_x^{(n)}\right\}$ in $\mathcal{F}$ for every $x \in V(a)$ such that $v_x^{(n)} \to h(x)$ as $n \to \infty$. By (1), there exists u_n in $\mathcal{F}$

such that $u_n \leq \min\left\{v_x^{(n)},\ x \in V(a)\right\}$. Then, $u_n(x) \to h(x)$ as $n \to \infty$ for every $x \in V(a)$. Let $v_n = P_a u_n$. Then, by (2), $v_n \in \mathcal{F}$ and $v_n \leq u_n$. Hence, $v_n(x) \to h(x)$ as $n \to \infty$ for every $x \in V(a)$; also v_n is q-harmonic at a. Consequently, we have $\Delta_q h(a) = \lim\limits_{n\to\infty} \Delta_q v_n(a) = 0$. That is, h is q-harmonic at a. □

Theorem 2.4.10. *(g.q-h.m.,greatest q-harmonic minorant) Let u (respectively v) be q-superharmonic (respectively q-subharmonic) on a subset F such that $v \leq u$. Then, there exists a q-harmonic function h on F such that $v \leq h \leq u$; moreover, if H is any other q-harmonic function on F such that $v \leq H \leq u$, then $H \leq h$. (Hence, h is called the greatest q-harmonic minorant of u.)*

Proof. Let $\mathcal{F}$ be the family of q-subharmonic functions t on F such that $t \leq u$. Then, $\mathcal{F}$ is an upper directed Δ_q-Perron family (of q-subharmonic functions) on $(F, \overset{0}{F})$ so that $h(x) = \sup t(x)$, where $t \in \mathcal{F}$, is a q-harmonic function on F such that $h \leq u$. Clearly, if H is a q-harmonic function on F dominated by u, then $H \in \mathcal{F}$. Consequently, $H \leq h$ on F. □

Theorem 2.4.11. *(Generalised q-Dirichlet Problem) Let F be a subset of X and $E \subset \overset{0}{F}$. Let f be a function on $F\backslash E$ such that $v \leq f \leq u$ on $F\backslash E$, where u and v are functions on F such that $u \geq v$ on F and $\Delta_q v \geq 0$ and $\Delta_q u \leq 0$ on E. Then there exists a function h on F such that $\Delta_q h = 0$ on E, $v \leq h \leq u$ on F and $h = f$ on $F\backslash E$. Moreover, if h_1 is any function on F such that $h_1 = f$ on $F\backslash E$ and $\Delta_q h_1 = 0$ on E, then $h_1 = h$ on F.*

Proof. Let $u_1 = f$ on $F\backslash E$ and $u_1 = u$ on E. Then $\Delta_q u_1 \leq 0$ on E. To see this, remark that if $x \in E$, then

$$
\begin{aligned}
[t(x) + q(x)]\, u_1(x) &= [t(x) + q(x)]\, u(x) \\
&\geq \sum_y t(x,y) u(y), \text{since } \Delta_q u(x) \leq 0, \text{by hypothesis} \\
&\geq \sum_y t(x,y) u_1(y), \text{since } u_1 \leq u \text{ on } V(E) \subset F.
\end{aligned}
$$

Hence, $\Delta_q u_1(x) \leq 0$ for every $x \in E$.

Similarly, if $v_1 = f$ on $F\backslash E$ and $v_1 = v$ on E, then $\Delta_q v_1 \geq 0$ at every vertex of E. Let $\mathcal{F}$ be the family of all real-valued functions s on F such that $v \leq v_1 \leq s \leq u_1 \leq u$ on F and $\Delta_q s \geq 0$ on E. Note that if s_1, s_2 are in $\mathcal{F}$, then $\Delta_q \sup(s_1, s_2) \geq 0$, so that $\sup(s_1, s_2) \in \mathcal{F}$; if $a \in E$, then

$$
P_a s(a) = \sum_z \frac{t(a,z)}{[t(a) + q(a)]} s(z) \geq \sum_z \frac{t(a,z)}{[t(a) + q(a)]} v_1(z) \geq v_1(a)
$$

and

$$P_a s(a) = \sum_z \frac{t(a,z)}{[t(a)+q(a)]} s(z) \le \sum_z \frac{t(a,z)}{[t(a)+q(a)]} u_1(z) \le u_1(a).$$

Hence, $P_a s \in \mathcal{F}$; finally, if $s \in \mathcal{F}$, then $s \le u$ on F. Consequently, $\mathcal{F}$ is an upper directed Δ_q-Perron family of q-subharmonic functions on (F, E). Hence, if $h(x) = \sup s(x)$, $s \in \mathcal{F}$, then $v_1 \le h \le u_1$ on F and $\Delta_q h = 0$ at every vertex of E; note that $h = f$ on $F \backslash E$ since $v_1 = u_1 = f$ on $F \backslash E$.

To prove the uniqueness of the solution h, suppose h_1 is another function such that $h_1 = f$ on $F \backslash E$, and $\Delta_q h_1 = 0$ at every vertex of E. Let $w = h - h_1$ on F. Then, w extended by 0 outside F satisfies the conditions: $\Delta_q w = 0$ on E and $w = 0$ on $X \backslash E$. Hence, by the Minimum Principle (Lemma 2.4.3), $w \equiv 0$ on X. □

Corollary 2.4.12. *(Classical q-Dirichlet problem) Let F be a proper subset of X. Let f be a function on ∂F. Then there exists a unique q-harmonic function h on F such that $h = f$ on ∂F.*

Proof. Suppose $|f| \le M$. Use the above theorem with $-M \le f \le M$ on ∂F and $E = \overset{0}{F}$, to get the unique solution. □

Theorem 2.4.13. *(q-Balayage) Let F be a proper subset of X and let u be a real-valued function on X such that $\Delta_q u \le 0$ at every vertex in $X \backslash F$. Then, there exists a unique function v on X such that $v \le u$ on X, $v = u$ on F, and $\Delta_q v = 0$ at every vertex in $X \backslash F$.*

Proof. By Theorem 2.4.6, there exists a unique function φ on X such that $\Delta_q \varphi = \Delta_q u = -(-\Delta_q)u$ at every vertex in $X \backslash F$, $\varphi = 0$ on F and $\varphi \ge 0$ on $X \backslash F$. Consequently, $\varphi \ge 0$ on X. Then, $v = u - \varphi$ on X has the stated properties. □

Remark 2.4.4. 1. The above q-balayaged function v can be described as the solution to an appropriate Dirichlet problem: Let $E = X \backslash F$. For some $M > 0$, let $-M < u$ on $V(E)$, where $V(E)$ stands for the union of E and all the neighbours of each vertex in E. Take $f = u$ on $V(E) \backslash E$. Since $\Delta_q(M) \le 0$ on X, we can (Theorem 2.4.11) construct the unique function h on $V(E)$ such that $h = f = u$ on $V(E) \backslash E$, $-M \le h \le u$ on $V(E)$, and $\Delta_q h = 0$ on E. Then, v is the function h on $V(E)$ extended by u on $X \backslash V(E)$.

2. In the context of an electrical network X, one can pose a problem similar to the capacity problem [27, p.52] in the classical potential theory, how to distribute a unit charge on a proper subset F so that the associated potential is constant on F. That is, the problem is to find $s \ge 0$ on X such that $(-\Delta)s \ge 0$ on X, $\sum_{x \in F} \Delta s(x) = -1$, $\Delta s(x) = 0$ if $x \in X \backslash F$, and for a constant α, $s(x) = \alpha$ if $x \in F$. This problem has no solution, since $(-\Delta)s \ge 0$ on a finite network X implies that s is a constant.

However, the above result on q-balayage shows that a solution to this problem exists when Δ is replaced by the operator Δ_q. For take $u \equiv 1$ on X. Then, $v \leq 1$ on X, $v = 1$ on F and $\Delta_q v = 0$ at every vertex in $X \backslash F$. Notice that $\Delta_q v \leq 0$ at every vertex in F. But $\Delta_q v(x) < 0$ for some x in F. For otherwise, v is q-harmonic on X and hence should be 0, a contradiction. Hence, $\beta = \sum_{x \in F} (-\Delta_q) v(x) > 0$. Write $\alpha = \beta^{-1}$ and $s(x) = \alpha v(x)$. Then, $\sum_{x \in F} \Delta_q s(x) = \alpha \sum_{x \in F} \Delta_q v(x) = -\alpha\beta = -1$, $\Delta_q s(x) = 0$ if $x \in X \backslash F$, and $s(x) = \alpha$ if $x \in F$.

The following theorem generalises the well-known *Condenser Principle* in finite electrical networks. In the context of finite networks with symmetric conductance, Bendito et al. [20] give a detailed form for this principle.

Theorem 2.4.14. *(Generalised q-Condenser Principle) Let A and B be two non-empty disjoint subsets of X. Let $F = X \backslash (A \cup B) \neq \phi$. Let a and b be two real numbers such that $a \leq 0 \leq b$. Then there exists a unique φ on X such that $a \leq \varphi(x) \leq b$ for every $x \in X$,*

$$\varphi(x) = a \text{ and } \Delta_q \varphi(x) \geq 0 \text{ for } x \in A,$$

$$\varphi(x) = b \text{ and } \Delta_q \varphi(x) \leq 0 \text{ for } x \in B,$$

$$\text{and } \Delta_q \varphi(x) = 0 \text{ for } x \in F.$$

Proof. In Theorem 2.4.6, take $f = 0$, $g(x) = a$ if $x \in A$, and $g(x) = b$ if $x \in B$. Then there exists a unique function φ on X, such that $\varphi = a$ on A, $\varphi = b$ on B, and $\Delta_q \varphi = 0$ on F (if $a = b = 0$, then $\varphi \equiv 0$). Note that by the Minimum Principle, $a \leq \varphi(x) \leq b$ on X. For, if $\psi(x) = \varphi(x) - a$ on X, then $\Delta_q \psi(x) = \Delta_q \varphi(x) + \Delta_q(-a)$. Since $(-a)$ is q-superharmonic, $\Delta_q(-a) \leq 0$. Hence, $\Delta_q \psi(x) \leq 0$ if $x \in F$, and $\psi(x) \geq 0$ if $x \notin F$. Hence (Lemma 2.4.3) $\psi \geq 0$ on X, that is $\varphi(x) \geq a$ on X. Similarly, considering the function $b - \varphi(x)$ on X, we conclude that $b - \varphi(x) \geq 0$ on X.

Consequently, at any vertex z where φ attains its minimum value a which is non-positive, $\Delta_q \varphi(z) \geq 0$; similarly, at every vertex y where φ attains its maximum value b which is non-negative, $\Delta_q \varphi(y) \leq 0$. Hence, $\Delta_q \varphi \geq 0$ on A and $\Delta_q \varphi \leq 0$ on B. □

Thus, φ is the q-Dirichlet solution as in Theorem 2.4.11, with F in the place of E and $V(F)$ in the place of F; $f = a$ on $A \cap \{V(F) \backslash F\}$, and $f = b$ on $B \cap \{V(F) \backslash F\}$; φ is extended by a on A, and by b on B.

Theorem 2.4.15. *(q-Green's function) Let F be a subset of X. If $y \in \overset{o}{F}$, then there exists a unique function $G_y^F(x) \geq 0$ on X such that $\Delta_q G_y^F(x) = -\delta_y(x)$ if $x \in \overset{0}{F}$, $G_y^F(s) = 0$ if $s \in X \backslash \overset{0}{F}$ and $G_y^F(x) \leq G_y^F(y)$ for any $x \in X$.*

Proof. Let $f(x) = -\delta_y(x)$ if $x \in \overset{0}{F}$. Then, by Theorem 2.4.6, there exists a unique function on X, noted by $G_y^F(x)$, such that $\Delta_q G_y^F(x) = f(x)$ if $x \in \overset{0}{F}$ and $G_y^F(s) = 0$ if $s \in X \setminus \overset{0}{F}$. Since $\Delta_q G_y^F(x) \leq 0$ on $\overset{0}{F}$ and $G_y^F = 0$ on $X \setminus \overset{\circ}{F}$ we conclude that $G_y^F \geq 0$ on X. (It can be seen that $G_y^F > 0$ on $\overset{0}{F}$ if $\overset{0}{F}$ is connected.)

To prove the last assertion, note that $G_y^F(x) = G_y^F(y)$ if $x = y$; if $x \neq y$, and $x \in \overset{0}{F}$, then $\Delta_q G_y^F(x) = 0$; and if $s \in X \setminus \overset{0}{F}$, then $G_y^F(s) = 0$ so that $\Delta_q G_y^F(s) \geq 0$, since $G_y^F \geq 0$ on X. Thus, applying the Domination Principle (Theorem 2.4.5, with $v(x) = G_y^F(y)$ for all $x \in X$, and $f(x) = G_y^F(x)$ for $x \in X$), we conclude that $G_y^F(x) \leq G_y^F(y)$ for all $x \in X$. □

Thus, $G_y^F(x)$ is the solution to the following Dirichlet problem: Let f be a function defined on $(X \setminus \overset{0}{F}) \cup \{y\}$ such that $f(y) = 1$ and $f(x) = 0$ if $x \in X \setminus \overset{0}{F}$. Then, from Theorem 2.4.11 (by taking X in the place of F and $\overset{0}{F} \setminus \{y\}$ in the place of E), we extend f onto X as the Dirichlet solution so that $\Delta_q f = 0$ on $\overset{0}{F} \setminus \{y\}$. By the Minimum Principle, $0 \leq f(x) \leq 1$ for all $x \in X$ and $\Delta_q f(y) < 0$. Then, $G_y^F(x) = \frac{f(x)}{(-\Delta_q) f(y)}$.

Remark 2.4.5. The above argument goes through even when $F = X$, so that for any $y \in X$, there exists a function $G_y(x) = G_y^F(x) > 0$ on X such that $\Delta_q G_y(x) = -\delta_y(x)$ and $G_y(x) \leq G_y(y)$ for all $x \in X$.

Proposition 2.4.16. *If $f(x)$is any function on X, then it is of the form $f(x) = \sum_y (-\Delta_q) f(y) G_y(x)$.*

Proof. Let $u(x) = \sum_y (-\Delta_q) f(y) G_y(x)$. Then, $(-\Delta_q) u(x) = (-\Delta_q) f(x)$ for all $x \in X$. Hence $u \equiv f$. □

Remark 2.4.6. The above proposition states that any real-valued function f on X is of the form $f = u - v$ on X, where $\Delta_q u \leq 0$ and $\Delta_q v \leq 0$.

Δ_q-Poisson kernel. Let F be a proper subset of X. $P_q(x, \xi)$ for $x \in F$ and $\xi \in \partial F$ is said to be the Δ_q-Poisson kernel of F, if it is the q-Dirichlet solution on F with boundary values $f(s) = \delta_\xi(s)$ for all $s \in \partial F$; note $P_q(s, \xi) = \delta_\xi(s)$ for s, ξ in ∂F and $\Delta_q P_q(x, \xi) = 0$ if $x \in \overset{0}{F}$ and $\xi \in \partial F$.

Proposition 2.4.17. *Let F be a proper subset of X. Then, the Dirichlet solution on F with boundary values $f(\xi)$ on ∂F is given by $h(x) = \sum_{\xi \in \partial F} P_q(x, \xi) f(\xi)$.*

Proof. We have $\Delta_q h(x) = 0$ if $x \in \overset{0}{F}$ and $h(s) = f(s)$ for any $s \in \partial F$. Hence, by the uniqueness of the q-Dirichlet solution, the proposition follows. □

Theorem 2.4.18. *Let F be a proper subset of X. Let u_n be a sequence of q-harmonic functions on F. Suppose $u_n(\xi) \to u(\xi)$ for every $\xi \in \partial F$. Then $\{u_n\}$ converges on F to a q-harmonic function h given by $h(x) = \sum_{\xi \in \partial F} P_q(x, \xi) u(\xi)$, if $x \in \overset{0}{F}$, and $h = u$ on ∂F.*

Proof. By Proposition 2.4.17, $h(x) = \sum_{\xi \in \partial F} P_q(x, \xi) u(\xi)$ is a q-harmonic function on F, being the Dirichlet solution on F with boundary values $u(\xi)$. By hypothesis, u_n is q-harmonic on F, so that $u_n(x) = \sum_{\xi \in \partial F} P_q(x, \xi) u_n(\xi)$, $x \in \overset{0}{F}$. Hence, $u_n(x) = \sum_{\xi \in \partial F} P_q(x, \xi) u_n(\xi) \to \sum_{\xi \in \partial F} P_q(x, \xi) u(\xi) = h(x)$. □

Chapter 3
Harmonic Function Theory on Infinite Networks

Abstract Similar to the classification of Riemannian manifolds as hyperbolic and parabolic, an infinite network X is said to be hyperbolic if the Green function is defined on X, otherwise X is said to be parabolic. Different criteria like Minimum Principle, the harmonic measure of the point at infinity and the sections determined by a vertex in X are introduced to effect the classification. The first part of this chapter studies the properties of superharmonic functions in hyperbolic networks: the existence of non-constant bounded or positive harmonic functions on X, domination principle and balayage. The second part carries out, in parabolic networks, the construction of superharmonic functions in X with point harmonic singularity, similar to the logarithmic potentials in the plane; such functions do not naturally appear in the context of random walks and electrical networks. Balayage, Maximum Principle and the representation of harmonic functions outside a finite set are considered in parabolic networks, leading to the concepts of flux at infinity and pseudo-potentials which play an important role in developing a potential theory on parabolic networks.

In the last chapter, we studied some properties of functions defined on finite graphs, by using matrices. Many of the useful problems in electrical networks and Markov chains are formulated as problems in finite graph theory. Yet, infinite graphs cannot be ignored. For example, since we do not know how to find solutions of many differential equations, we try to obtain their approximate solutions by various means. One such method is to solve partial differential equations by the finite difference approximation, which involves horizontal and vertical displacements. This in effect resembles an electrical grid which is a graph. However, for some differential equations, like wave equations, the domain of existence of the solution may be unbounded, suggesting a problem in a graph with infinite vertices and consequently a need to consider situations like infinite electrical grids [73]. A similar connection exists between infinite electrical networks and Markov chains [68].

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As a consequence of the inherent similarities between the basic structures of infinite electrical networks and Markov chains, some problems in Markov chains can be solved by the methods developed in the context of electrical networks and in the other direction some problems in the context of electrical networks can be more easily solved by probabilistic methods. Hence a common approach to study these two subjects is desirable. This raises the possibility of introducing an abstract structure, called simply an infinite network whose theoretical developments, problems and methods of solutions will have some resemblances to and some influences from infinite electrical networks and Markov chains.

3.1 Infinite Networks and the Laplace Operator

By an infinite network X, we mean a graph with a countably infinite number of vertices and a countably infinite number of edges with the following properties:

1. X is connected. That is, given any two vertices a and b in X, there exists a finite path $\{a = a_0, a_1, ..., a_n = b\}$ connecting a and b. We say that the length of this path is n.
2. X is locally finite. That is, for any vertex x, the number of neighbours of x is finite. If x and y are neighbours, then we write $x \sim y$.
3. X has no self-loop. That is, there is no edge from a to a.
4. For any pair of vertices x and y, there is an associated number $t(x, y) \geq 0$ such that $t(x, y) > 0$ if and only if $x \sim y$. Consequently, note that for any vertex x, $\sum\limits_{y \in X} t(x, y) = t(x) > 0$. There are only a finite number of non-zero terms in the summation. (Note that we do not place the restriction $t(x, y) = t(y, x)$. If $t(x, y) = t(y, x)$ for every pair of vertices in X, then we say that X has symmetric conductance.)

As defined earlier, let us say that for a subset E of the infinite network X, x is an *interior vertex* if and only if x and all its neighbours are in E. The set of all interior points of E is denoted by $\overset{0}{E}$ and the *boundary* of E is $\partial E = E \backslash \overset{0}{E}$. Let u be a real-valued function defined on E. For $x \in \overset{0}{E}$, let us define the Laplacian of u at x as $\Delta u(x) = \sum\limits_{y} t(x, y)[u(y) - u(x)]$. u is said to be *superharmonic* on E, if and only if $\Delta u(x) \leq 0$ for every $x \in \overset{0}{E}$; u is *subharmonic* on E if and only if $\Delta u(x) \geq 0$ for every $x \in \overset{0}{E}$; and u is *harmonic* on E if and only if $\Delta u(x) = 0$ for every $x \in \overset{0}{E}$. The following properties of superharmonic functions can be derived as in the previous chapter:

1. If s_1 and s_2 are superharmonic on a subset E and if α_1, α_2 are two non-negative numbers, then $\alpha_1 s_1 + \alpha_2 s_2$ and $\inf(s_1, s_2)$ are superharmonic on E.
2. If $\{s_n\}$ is a sequence of superharmonic functions on E such that $s(x) = \lim\limits_{n} s_n(x)$ is finite at every $x \in E$, then s is superharmonic on E.

3. If s is a superharmonic function on a finite subset E, then $\min_{x \in E} s(x) = \min_{z \in \partial E} s(z)$. As a consequence, if h is harmonic on a finite subset E and if $h = 0$ on ∂E, then $h \equiv 0$ on E.
4. Harnack Property: Let F be a subset of X, and E be a connected subset of $\overset{0}{F}$. Let a and b be two vertices in E. Then, there exist two constants $\alpha > 0$ and $\beta > 0$ such that for any non-negative superharmonic function s on F, $\alpha s(b) \leq s(a) \leq \beta s(b)$.

Proof. Since E is connected, there exists a path $\{a = a_0, a_1, ..., a_n = b\}$ connecting a and b in E. Take any non-negative superharmonic function s on F. Then, $t(a)s(a) \geq \sum_{x \sim a} t(a,x)s(x)$. In particular, $t(a)s(a) \geq t(a,a_1)s(a_1)$. Again $t(a_1)s(a_1) \geq \sum_{x \sim a_1} t(a_1,x)s(x)$, so that $t(a_1)s(a_1) \geq t(a_1,a_2)s(a_2)$. Hence

$$s(a) \geq \frac{t(a,a_1)}{t(a)} \times \frac{t(a_1,a_2)}{t(a_1)} s(a_2).$$

Proceeding further, we arrive at the inequality

$$s(a) \geq \frac{t(a,a_1)}{t(a)} \times \frac{t(a_1,a_2)}{t(a_1)} \times ... \times \frac{t(a_{n-1},a_n)}{t(a_{n-1})} s(b),$$

which is of the form $s(a) \geq \alpha s(b)$. The other inequality $s(a) \leq \beta s(b)$ is proved similarly. Note that α, β do not depend on the choice of the superharmonic function s.

5. If s is superharmonic on E and v is subharmonic on E such that $s \geq v$, then there exists a harmonic function h on E such that $s \geq h \geq v$ on E; this function h can be chosen such that if u is another harmonic function on E such that $s \geq u \geq v$, then $h \geq u$. h is called the greatest harmonic minorant (g.h.m.) of s on E.

   Using a Perron family, we proved a similar result in Theorem 2.4.10 to construct the greatest q-harmonic minorant of a q-superharmonic function defined on a finite set F. The same method works also in an infinite network X for the construction of the greatest harmonic minorant of s on an arbitrary set E in X.
6. Maximum Principle: Let F be a finite subset in an infinite network X. Suppose u is defined on X such that $\Delta u(x) \geq 0$ at every vertex in F and $u \leq 0$ on $X \backslash F$. Then $u \leq 0$ on X.

   For, suppose $\sup_{x \in F} u(x) = M > 0$. Then $u(x_0) = M$ for some $x_0 \in F$. If $x_0 \in \partial F$, there is at least one vertex $z \in X \backslash F$ such that $x_0 \sim z$. Then,

$$0 \leq \Delta u(x_0) = \sum_{x \sim x_0} t(x_0, x)[u(x) - u(x_0)] < 0,$$

since $u(x) - u(x_0) \leq 0$ for all x and $u(z) - u(x_0) < 0$. This contradiction shows that $M \leq 0$.

If $x_0 \notin \partial F$, then $x_0 \in \overset{0}{F}$. Take a vertex $y \in X \backslash F$. Since X is connected, there is a path $\{x_0, x_1, ..., x_n = y\}$ connecting x_0 to y. Let $i \geq 1$ be the largest index such that $x_j \in \overset{0}{F}$ if $j \leq i - 1$. Then $x_i \in \partial F$ since $x_i \sim x_{i-1} \in \overset{0}{F}$. Note $i \leq n - 1$ since $x_n = y \notin F$. Also, $u(x_k) = M$ for $0 \leq k \leq i$, which comes from the fact that by induction, for $j \leq i - 1$,

$$0 \leq \Delta u(x_j) = \sum_{x \sim x_j} t(x_j, x)[u(x) - u(x_j)] \leq 0.$$

Since $x_i \in \partial F$, there exists some vertex $a \in X \backslash F$ such that $a \sim x_i$. Then

$$0 \leq \Delta u(x_i) = \sum_{x \sim x_i} t(x_i, x)[u(x) - u(x_i)] < 0,$$

since $u(x) - u(x_i) \leq 0$ for all x and $u(a) - u(x_i) \leq 0 - M < 0$. This contradiction shows that $M \leq 0$, that is $u \leq 0$ on X.

7. We shall frequently use the above maximum principle in the form: Let F be a finite subset of an infinite network X. Let u be a real-valued function on X such that $\Delta u = 0$ at every vertex in F and $u = 0$ on $X \backslash F$. Then $u \equiv 0$.

Definition 3.1.1. A non-negative superharmonic function p defined on a subset E is said to be a potential if and only if the greatest harmonic minorant (g.h.m.) of p on E is 0.

Alternatively, let $s \geq 0$ be a superharmonic function on E. Suppose v is an arbitrary subharmonic function on E. Then, s is a potential if and only if $s \geq v$ implies that $v \leq 0$. Some of the properties of potentials are as follows:

1. If p_1 and p_2 are potentials on E and if α_1 and α_2 are non-negative numbers, then $\alpha_1 p_1 + \alpha_2 p_2$ is a potential on E.

   First, $\alpha_1 p_1 + \alpha_2 p_2$ is a non-negative superharmonic function on E. If $\alpha_1 = 0 = \alpha_2$, then nothing to prove. Let $\alpha_1 > 0$ and v be a subharmonic function on E such that $v \leq \alpha_1 p_1 + \alpha_2 p_2$. Then, the subharmonic function $\frac{v - \alpha_2 p_2}{\alpha_1}$ is majorized by the potential p_1. Hence $v - \alpha_2 p_2 \leq 0$. If $\alpha_2 = 0$, then $v \leq 0$. Otherwise, $\frac{v}{\alpha_2} \leq p_2$ and hence $v \leq 0$. Consequently, $\alpha_1 p_1 + \alpha_2 p_2$ is a potential on E.
2. If $s \geq 0$ is superharmonic on E and if p is a potential on E such that $s \leq p$, then s is a potential on E.

   For, if v is subharmonic on E and $v \leq s$, then v is majorized by the potential p. Hence, $v \leq 0$.

3. Let $p(x) = \sum_{n\geq 1} p_n(x) < \infty$, for every $x \in E$, where each p_n is a potential on E. Then, p is a potential on E.

   First note that p is superharmonic on E; to prove that p is actually a potential, let $v \leq p$ where v is subharmonic on E. Then $v(x) - \sum_{n\geq 2} p_n(x) \leq p_1(x)$. Here the left side is subharmonic and the right side is a potential, so that $v - \sum_{n\geq 2} p_n \leq 0$. By a similar argument, for any integer k, $v \leq \sum_{n\geq k} p_n$. Let x be an arbitrary vertex in E. Since $\sum_{n\geq 1} p_n(x)$ is convergent, for m large $\sum_{n\geq m} p_n(x) \leq \varepsilon$. Hence, $v(x) \leq \varepsilon$. Since ε is arbitrary, $v(x) \leq 0$. This means that p is a potential on E.

   Caution: Let p_n be a sequence of potentials on E and let $p(x) = \lim_n p_n(x)$ be finite for each $x \in E$. Even then, we cannot say that p is a potential on E. For, we shall show later that any superharmonic function $s \geq 0$ on E is an increasing limit of potentials on E.
4. Riesz representation: Any superharmonic function $s \geq 0$ on E can be written as the sum of a potential and a non-negative harmonic function on E; and this representation is unique.

   For, if h is the g.h.m. of s on E, then $s \geq h \geq 0$ so that the g.h.m. of the non-negative superharmonic function $p = s - h$ is 0. Hence p is a potential and $s = p + h$ on E. Suppose $s = p_1 + h_1$ is another such representation, then $p = p_1 + h_1 - h \geq h_1 - h$. Since p is a potential majorizing the harmonic function $h_1 - h$, we conclude $h_1 - h \leq 0$. Similarly, we prove $h - h_1 \leq 0$. Hence, $h = h_1$ and consequently, $p = p_1$.

   We have defined potentials on E. But it is not obvious that on any subset E of an infinite network X, there exist positive potentials. In fact, there are networks without having any positive potential defined on them.

Definition 3.1.2. An infinite network X is said to be a hyperbolic network if and only if there exists a positive potential on X; otherwise, X is called a parabolic network.

Proposition 3.1.1. *If there exists a non-constant positive superharmonic function on X, then X is hyperbolic.*

Proof. Let $s > 0$ be a non-constant superharmonic function on X. (Recall that a harmonic function is both superharmonic and subharmonic.) For some $a \in X$, let $u(x) = \frac{s(x)}{s(a)}$, which is a non-constant superharmonic function. Let $v(x) = \inf(u(x), 1)$. Then, we know that v is superharmonic on X, but it cannot be harmonic. For, if v were harmonic, then v attains its maximum value at $x = a$, and hence by the Maximum Principle $v \equiv 1$. This implies that the superharmonic function $u \geq 1$, attains its minimum at $x = a$ so that u should be a constant, a contradiction. Consequently, v being (non-harmonic) positive superharmonic, in the Riesz representation $v = p + h$, we should have $p > 0$. Hence, X is hyperbolic. □

Corollary 3.1.2. *X is parabolic if and only if every positive superharmonic function on X is a constant.*

Proof. If every positive superharmonic function is constant on X, there cannot be any positive potential on X, hence X is parabolic. Conversely, if there exists a non-constant positive superharmonic function on X, then X is hyperbolic by the above theorem. □

We have given in Corollary 1.4.3 a Minimum Principle for superharmonic functions on a set E consisting of only a finite number of vertices. If E is not necessarily finite, we need certain modifications.

Proposition 3.1.3. *Let X be hyperbolic. Let E be an arbitrary proper subset of X. Suppose v is superharmonic on E such that $v \geq 0$ on ∂E and $v \geq -p$ on E, where p is a potential on X. Then, $v \geq 0$ on E.*

Proof. Let $u = \inf(v, 0)$ in E extended by 0 outside E. Then, u is superharmonic on X and $u \geq -p$ on X. Since p is a potential majorizing the subharmonic function $-u$ on X, we conclude that $-u \leq 0$ on X. In particular, on E, $u \geq 0$ and hence $v \geq 0$. □

Corollary 3.1.4. *Let X be hyperbolic. Let E be an arbitrary set of X. Suppose h is harmonic on E such that $h = 0$ on ∂E and $|h| \leq p$ on E, where p is a potential on X. Then, $h \equiv 0$.*

Proof. Since $h \geq -p$ and $-h \geq -p$ on E and $h = 0$ on ∂E, by the above proposition, $h \equiv 0$. □

Proposition 3.1.5. *Let X be parabolic. Let E be an arbitrary proper subset of X. Suppose v is a lower bounded superharmonic function on E and $v \geq 0$ on ∂E. Then, $v \geq 0$ on E.*

Proof. Let $u = \inf(v, 0)$ on E extended by 0 outside E. Then, u is superharmonic on X and lower bounded too. Hence, u is constant since X is parabolic. That is, $u \equiv 0$ so that $v \geq 0$ on E. □

Corollary 3.1.6. *Let X be parabolic. Let E be an arbitrary subset of X. Suppose h is a bounded harmonic function on E. If $h = 0$ on ∂E, then $h \equiv 0$.*

Proof. This is a simple consequence of the above proposition. □

Recall that Theorem 2.4.11 has been proved without any reference to whether the network X is finite or not, by using a Perron family of superharmonic functions. We shall state that theorem here.

Theorem 3.1.7. *(Generalised Dirichlet Problem for infinite networks) Let F be a subset of an infinite network X and $E \subset \overset{0}{F}$. Suppose f is a function defined on $F \backslash E$ such that $v \leq f \leq u$ on $F \backslash E$ where u and v are defined on F, $v \leq u$ on F, $\Delta u \leq 0$ and $\Delta v \geq 0$ at each vertex in E. Then, there exists a function h on F such that $v \leq h \leq u$ on F, $h = f$ on $F \backslash E$, and $\Delta h(x) = 0$ at each $x \in E$, and the* *function h can be so chosen that if h_1 is another such harmonic function on F with these three properties, then $h_1 \leq h$.*

Proof. Let $u_1 = f$ on $F \backslash E$ and $u_1 = u$ on E. Let $v_1 = f$ on $F \backslash E$ and $v_1 = v$ on E. Then, as shown in the proof of Theorem 2.4.11, $\Delta u_1(x) \leq 0$ and $\Delta v_1(x) \geq 0$ at every vertex in E. Let $\Im$ be the family of real-valued functions s on F such that $v \leq v_1 \leq s \leq u_1 \leq u$ on F and $\Delta s \geq 0$ on E. Then $\Im$ is a Perron family of subharmonic functions on E (see the proof of Theorem 2.4.11). Hence, if $h(x) = \sup s(x)$, $x \in F$, $s \in \Im$, then $v \leq h \leq u$ on f, $h = f$ on $F \backslash E$, and $\Delta h = 0$ at every vertex in E.

Suppose h_1 is another function on f such that $v \leq h_1 \leq u$ on F, $h_1 = f$ on $F \backslash E$, and $\Delta h_1(x) = 0$ at each vertex x in E. Then, $h_1 \in \Im$ and consequently, $h_1 \leq h$. □

Corollary 3.1.8. *Let X be hyperbolic. Suppose E is a subset of X. Let f be a function on ∂E such that $|f| \leq p$ on ∂E where p is a potential on X. Then, there exists a unique harmonic function h on E such that $h = f$ on ∂E and $|h| \leq p$ on E.*

Proof. The existence follows from the above theorem and the uniqueness is from Corollary 3.1.4. □

Corollary 3.1.9. *Let X be parabolic. Suppose E is a subset of X. Let f be a bounded function defined on ∂E. Then there exists a unique bounded harmonic function h on E such that $h = f$ on ∂E.*

Proof. The existence of a harmonic function on E such that $h = f$ on ∂E is a consequence of the above theorem. For the uniqueness, suppose h_1 is another such function. Write $\varphi = h_1 - h$, so that φ is bounded harmonic on E with boundary value 0. Hence, by Corollary 3.1.6, $\varphi \equiv 0$. □

Theorem 3.1.10. *(Balayage) Let s be a non-negative superharmonic function on a network X. Let A be a proper subset of X. Then, there exists a non-negative superharmonic function R_s^A on X such that $R_s^A \leq s$ on X, $R_s^A = s$ on A, and $\Delta R_s^A(x) = 0$ if $x \in X \backslash A$. If there exists a potential p on X such that $s \leq p$ on A, then R_s^A is a potential on X.*

Proof. In Theorem 3.1.7 (Dirichlet problem) take $E = X \backslash A$, $F = V(E)$ which is the set consisting of E and all the neighbours of each vertex in E, $f = s$, $u = s$, and $v = 0$. Then, there exists a function h on F such that $\Delta h(x) = 0$ if $x \in E$, $h = s$ on $F \backslash E$, and $0 \leq h \leq s$ on F. Let R_s^A denote the function h on F extended by s on $X \backslash F$. Then, R_s^A is a non-negative superharmonic function on X, having the properties stated in the theorem.

Suppose p is a potential on X such that $s \leq p$ on A. Then in the above paragraph, replace s by $s_1 = \inf(s, p)$ which is a potential on X. Hence $R_{s_1}^A$ is a potential since it is majorized by the potential s_1. Take R_s^A as the potential $R_{s_1}^A$. Since $s_1 = s$ on A, we notice that R_s^A has the stated properties, in addition to being a potential on X. □

Remark 3.1.1. If X is parabolic, then the non-negative superharmonic function s is constant and we take $R_s^A = s$ on X. Hence the balayage in this form in a parabolic network is not important. Theorem 3.4.4 gives a different version of balayage in a parabolic network.

3.2 Classification of Infinite Networks

We have so far seen two classes of infinite networks, hyperbolic and parabolic. In this section, we shall carry out further classifications of networks so that certain unified results in each class can be formulated. Let us fix a vertex e. For each $e_i \sim e$, we shall refer to the subset $[e, e_i] = \{x$: there exists a path joining x to e which passes through $e_i\}$ as *the section determined by e and* e_i. It is assumed that e and e_i are in $[e, e_i]$. Then, $[e, e_i]$ and $[e, e_i]\setminus \{e\}$ are connected, may be a finite set. Since X is an infinite set and since e has only a finite number of neighbours, then at least one $[e, e_i]$ should contain an infinite number of vertices. $[e, e_i]$ is referred to as an *infinite section* if there are infinite vertices in it; otherwise $[e, e_i]$ is a *finite section*. If e_i and e_j are two neighbours of e, then either $[e, e_i]$ and $[e, e_j]$ are two sections having e as the only common vertex or $[e, e_i] = [e, e_j]$. If $E_i = [e, e_i]$, then each $x \in E_i\setminus \{e\}$ is an interior vertex of E_i. For, if $x \in E_i\setminus \{e\}$ and if $y \sim x$, then $y \in E_i$.

Proposition 3.2.1. *Let $E_i = [e, e_i]$ be a finite section. Let u be a function defined on E_i such that $u(e) = 0$ and $\Delta u(x) = 0$ at each $x \in E_i\setminus \{e\}$. Then $u \equiv 0$.*

Proof. Suppose u takes positive values on E_i. Then $\max_{x\in E_i} u(x) = M > 0$. Let $u(y) = M$ for $y \in E_i$. As $\Delta u(x) = 0$ for each $x \in E_i\setminus \{e\}$, $u(z) = M$ at all neighbours z of y. Since the vertex y is connected to the vertex e, by induction, we should have $u(e) = M > 0$, a contradiction. Hence, $u \leq 0$ on E_i; and similarly we prove $u \geq 0$ on E_i. Thus, $u \equiv 0$. □

Theorem 3.2.2. *Let $E_i = [e, e_i]$ be an infinite section. Then, there exists $h \geq 0$ on E_i such that $h(e) = 0$, $h(x) > 0$ if $x \in E_i\setminus \{e\}$ and $\Delta h(x) = 0$ at every vertex $x \in E_i\setminus \{e\}$.*

Proof. For $z \in X$, let $|z|$ denote the shortest distance of z from e. Let $B_n = \{x : |x| \leq n\}$. Let $B_n^{'} = B_n \cap [e, e_i]$. Then, by using the Dirichlet solution, there is a function u_n on $B_n^{'}$ such that $0 \leq u_n \leq 1$, $u_n(e) = 0$, and $u_n(y) = 1$ if $y \in \partial B_n^{'}\setminus \{e\}$, $u_n > 0$ on $\overset{0}{B_n^{'}}\setminus \{e\}$, and $\Delta u_n(x) = 0$ if $x \in \overset{0}{B_n^{'}}\setminus \{e\}$. Assume u_n is extended to the rest of E_i by giving the value $u_n(a) = 1$ for each $a \in E_i\setminus B_n^{'}$. Clearly, u_n is a superharmonic function on E_i. Fix a vertex , $a \neq e$, and define $v_n(x) = \frac{u_n(x)}{u_n(a)}$ on E_i. Then, $v_n \geq 0$ is a superharmonic function on E_i, $v_n(a) = 1$, $v_n(e) = 0$ and $v_n > 0$ on $E_i\setminus \{e\}$.

Take a vertex b in $E_i \setminus \{e\}$. Then, by the Harnack property, there is an $\alpha > 0$ such that $v_n(b) \leq \alpha v_n(a) = \alpha$ for all n. Hence, we can extract a subsequence $\left\{v_n^{'}\right\}$ from $\{v_n\}$ such that $\lim_{n\to\infty} v_n^{'}(b)$ exists. Take another vertex k in $E_i \setminus \{e\}$. Then, by using the same argument, we can extract a subsequence $\left\{v_n^{''}\right\}$ from $\left\{v_n^{'}\right\}$ such that $\lim_{n\to\infty} v_n^{''}(k)$ exists. Note that $\lim_{n\to\infty} v_n^{''}(b)$ also exists. Since E_i is countably infinite, this method shows that there exists a subsequence $\{v_n^*\}$ of $\{v_n\}$ such that $h(x) = \lim_{n\to\infty} v_n^*(x)$ exists for each $x \in E_i \setminus \{e\}$.

Since $v_n^* \geq 0$ is superharmonic on E_i, $v_n^*(e) = 0$ and $v_n^*(a) = 1$, we find $h(x)$ is superharmonic on E_i, $h(e) = 0$ and $h(a) = 1$. Hence, $h(x) > 0$ if $x \in E_i \setminus \{e\}$. Finally, at any vertex z in $E_i \setminus \{e\}$, $\Delta v_n^*(z) = 0$ for all n sufficiently large, so that $h(x)$ is harmonic at z. □

Notation: In an infinite network X, let e be a vertex. Denote by $H_0^+(e)$ the set of non-negative functions in X such that $h(e) = 0$ and $\Delta h(x) = 0$ if $x \neq e$. Note that $H_0^+(e)$ contains at least one function h not identically 0. For, if E is an infinite section determined by e, define h on E such that $h(e) = 0$ and $h(x) > 0$, $\Delta h(x) = 0$ if $x \neq e$. This function h extended by 0 outside E is in $H_0^+(e)$.

Definition 3.2.1. Let $E_i = [e, e_i]$ be an infinite section. Then, by the above theorem, there always exists at least one function $h \geq 0$ on E_i such that $h(e) = 0$ and for $x \in E_i \setminus \{e\}$, $\Delta h(x) = 0$ and $h(x) > 0$. E_i is called a P-section if one such function h is bounded. Otherwise, E_i is called an S-section, that is in an S-section every such function h is unbounded.

Example: Let T be a standard homogeneous tree of degree $q + 1$, $q \geq 2$, that is every vertex in T has exactly $q + 1$ neighbours and $t(x, y) = t(y, x) = \frac{1}{q+1}$ for any pair $x \sim y$. Let e be any fixed vertex. Then, there are $q + 1$ sections determined by e, each of which is infinite. Measuring distances from e, define $h(e) = 0$, $h(e_i) = 1$ if $e_i \sim e$, and $h(x_n) = \sum_{k=0}^{n} q^{-k}$ if $|x_n| = n + 1$. Then, $\Delta h(e_i) = (0 - 1) + q\left[(1 + \frac{1}{q}) - 1\right] = 0$; and if $x \sim e_i$, $x \neq e$, then $\Delta h(x) = \left[1 - (1 + \frac{1}{q})\right] + q\left[(1 + \frac{1}{q} + \frac{1}{q^2}) - (1 + \frac{1}{q})\right] = 0$. Now, for $n > 1$, x_n has one neighbour y with $|y| = n$ and q neighbours z_j with $|z_j| = n + 2$. Hence,

$$\begin{aligned}\sum_{x_n \sim x} t(x_n, x)h(x) &= t(x_n, y)h(y) + \sum_{j=1}^{q} t(x_n, z_j)h(z_j) \\ &= \frac{1}{q+1} \sum_{0}^{n-1} q^{-k} + \frac{q}{q+1} \sum_{0}^{n+1} q^{-k} \\ &= \frac{1}{q^n} \times \frac{q^{n+1} - 1}{q - 1} = h(x_n).\end{aligned}$$

That is, h is harmonic at all vertices in $E_i \setminus \{e\}$. Since $q \geq 2$, h is bounded. Hence E_i is a P-section.

But on a P-section, there can be unbounded positive harmonic functions with the above properties. For example, when $q = 2$, let $[e, e_1]$ be a section. Define $h \geq 0$ on $[e, e_1]$ such that $h(e) = 0, h(e_1) = 1$; if x_{11}, x_{12} are the other neighbours of e_1, let $h(x) = 1$ if $x = x_{12}$ or a descendant of x_{12}. Take $h(x_{11}) = 2$. Let x_{21}, x_{22} be the other neighbours of x_{11}. Let $h(x) = 2$ if $x = x_{21}$ or a descendant of x_{21}. Take $h(x_{22}) = 3$ and so on. Then $h \geq 0$ is harmonic at each vertex in $[e, e_1] \setminus \{e\}$, $h(e) = 0, h(e_1) = 1$ and $h(x_{nn}) = n+1$ if $n \geq 1$. Thus on the P-section $[e, e_1]$, an unbounded harmonic function $h \geq 0$ is defined such that $h(e) = 0$ and $h(e_1) = 1$.

Remark 3.2.1. If $[e, e_1]$ is an infinite section and if $h \geq 0$ is an unbounded harmonic function on $[e, e_1]$ such that $h(e) = 0$ and $h(e_1) = 1$, then the above example shows that it is not possible to decide whether $[e, e_1]$ is a P-section or an S-section, based on the function h. Hence Theorem 3.2.3 gives a better way to characterize P-sections. However, Definition 3.2.1 gives a more practical way of distinguishing P-sections from S-sections.

Example: (a) Let $X = \{e, e_1, e_2, ...\}$ be a linear network, with $t(x, y) = \frac{1}{2}$ for any pair $x \sim y$. Then, $E = [e, e_1] = X$ is an infinite section. Suppose, $h \geq 0$ is defined on E such that $h(e) = 0$, $h(e_1) = 1$ and $\Delta h(x) = 0$ for every $x \in E \setminus \{e\}$. Then, $h(e_n) = n$ for all $n \geq 2$. Hence, E is an S-section.

(b) Let X consist of two infinite rays $\{e, e_1, x_1, x_2, ...\}$ and $\{e, e_2, y_1, y_2, ...\}$ with a common vertex e. Assume $e_1 \sim e_2$ and $t(x, y) = \frac{1}{2}$ for any pair of neighbouring vertices x and y. Then, $[e, e_1] = [e, e_2] = X$. Let $h \geq 0$ be defined on X such that $h(e) = 0$ and $\Delta h(x) = 0$ for every $x \in X \setminus \{e\}$. Suppose $h(e_1) = a$ and $h(e_2) = b$. Then, using the property $\Delta h(x) = 0$ for $x \in X \setminus \{e\}$, we calculate $h(x_n) = (2n+1)a - nb$, and $h(y_n) = (2n+1)b - na$, for $n \geq 1$. The properties of h then show that h cannot be bounded, that is $[e, e_1]$ is not a P-section. If $a = b = 1$, then we get an unbounded non-negative harmonic function h on $[e, e_1]$ such that $h(e) = 0$ and $\Delta h(x) = 0$ if $x \neq e$.

3.2.1 Harmonic Measure at Infinity of a Section

In $\Re^n, n \geq 2$, when we solve the Dirichlet problem in $|x| > 1$with boundary values 0 on $|x| = 1$ and 1 at the Alexandroff point at infinity, the solution h is identically 0 when $n = 2$ and $h > 0$ when $n > 2$. We then say that the harmonic measure of the point at infinity is 0 in $\Re^2$ and positive in $\Re^n, n \geq 3$. This distinction characterizes the profound difference in the study of potential theory in $\Re^2$ on the one hand and in $\Re^n, n \geq 3$, on the other. (See, for example, [27] and pp.103–112 in [5]) This is also a basis for the classification of Riemann surfaces as parabolic and hyperbolic. An analogous development in the discrete case of a network can be carried out as follows.

Let $E = [e, e_1]$ be an infinite section. As in the proof of Theorem 3.2.2 , let u_n be the function defined on E with the properties: $0 \leq u_n \leq 1$ on E, $u_n(e) = 0$, $u_n(y) = 1$ if $y \in \partial B_n^{'} \setminus \{e\}$ where $B_n^{'} = B_n \cap [e, e_1]$, $u_n(a) = 1$ if $a \in E \setminus B_n^{'}$ and $\Delta u_n(x) = 0$ if $x \in \overset{0}{B_n^{'}} \setminus \{e\}$. With these properties, $u_n(y) = 1$ and $\Delta u_n(y) \leq 0$ if $y \in \partial B_n^{'} \setminus \{e\}$. Moreover,$\{u_n\}$ is a decreasing sequence in E. Let $h(x) = \lim_{n \to \infty} u_n(x)$, $x \in E$. Then, $0 \leq h(x) \leq 1$ if $x \in E$, $h(e) = 0$ and $\Delta h(y) = 0$ if $y \in E \setminus \{e\}$. By the Minimum Principle, either $h \equiv 0$ or $0 < h < 1$ on $E \setminus \{e\}$. We say that the *harmonic measure at infinity of the section* E is 0 if $h \equiv 0$; otherwise the harmonic measure at infinity of the section E is said to be *positive*.

Theorem 3.2.3. *Let $E = [e, e_1]$ be an infinite section in a network X. Then, E is a P-section if and only if the harmonic measure at infinity of the section E is positive.*

Proof. Let E be a P-section. Then, by definition, there exists a bounded function u on E such that $0 \leq u \leq M$, $u(e) = 0$ and $\Delta u(x) = 0$ if $x \in E \setminus \{e\}$. Let $v(x) = \frac{u(x)}{M}$. Then, by the Maximum Principle, $u_n(x) \geq v(x)$ on $B_n^{'}$ and consequently, $u_n(x) \geq v(x)$ on E, which implies that $h(x) = \lim_{n \to \infty} u_n(x) \geq v(x)$ on E. That is, the harmonic measure at infinity of the section E is positive.

Conversely, if the harmonic measure at infinity of E is positive, then there exists a function h on E such that $0 < h(x) < 1$ and $\Delta h(x) = 0$ if $x \in E \setminus \{e\}$ and $h(e) = 0$. The existence of such a function in E means that E is a P-section. □

Theorem 3.2.4. *A network X is hyperbolic if and only if a vertex e in X determines at least one P-section.*

Proof. Let $E = [e, e_1]$ be a P-section determined by the vertex e. That is, there exists a bounded function $h \geq 0$ on E, such that $h(e) = 0$, $h(x) > 0$ and $\Delta h(x) = 0$ for every $x \in E \setminus \{e\}$. Extend h to X by giving the value 0 for the vertices outside E. Then, $h \geq 0$ is a non-constant bounded function on X such that $\Delta h(y) \geq 0$ whenever $h(y) = 0$. Consequently, $(-h)$ is a bounded non-constant superharmonic function on X. Hence (Proposition 3.1.1), there exists a positive potential on X, that is X is a hyperbolic network.

Conversely, let X be a hyperbolic network. Let $p > 0$ be a potential on X. Let $q(x) = \inf \left(\frac{p(x)}{p(e)}, 1 \right)$. Then $1 \geq q$ is a potential on X, $q(e) = 1$. If $q \equiv 1$ at each one of the infinite sections determined by the vertex e, then $q \equiv 1$ outside a finite set in X. Then, by the Minimum Principle, $q \geq 1$ on X. This means that $q \equiv 1$ on X, which implies that $p(x)$ attains its minimum at $x = e$ and hence constant, not true. Hence, there is an infinite section $E = [e, e_1]$ in which $q(y) < 1$ for some y in $E \setminus \{e\}$.

Now in Theorem 3.1.7 (Dirichlet solution), take $F = X$ so that $\overset{0}{F} = X$ and $X \setminus \{e\} \subset \overset{0}{F}$. Take $f(x) = 1$ when $x = e$, $u(x) = q(x)$, and $v \equiv 0$. Then, there exists a function $h(x)$ on X such that $0 \leq h(x) \leq q(x) \leq 1$, $h(e) = 1$ and $\Delta h(x) = 0$ if $x \neq e$. Note that, $\Delta h(e) < 0$. Hence, h is a non-negative superharmonic function on X, majorized by the potential q, so that h itself is a potential on X. Since $h(e) > 0$, h is a positive potential on X. Now, $h(y) \leq q(y) < 1$ so that $h(x) < 1$ for $x \in E \setminus \{e\}$; for, if $h(z) = 1$ at some $z \in E \setminus \{e\}$ which is connected, then by the Maximum Principle $h \equiv 1$ on $E \setminus \{e\}$, a contradiction. Consequently, when $x \in E \setminus \{e\}$, $0 < h(x) < 1$ and $\Delta h(x) = 0$ and $h(e) = 1$. Define $h'(x) = 1 - h(x)$ on $E \setminus \{e\}$ Then, $0 \leq h'(x) < 1$ on $E \setminus \{e\}$, $h'(e) = 0$, and when $x \in E \setminus \{e\}$, $h'(x) > 0$ and $\Delta h'(x) = 0$. That is, E is a P-section. □

Lemma 3.2.5. *Suppose $\{E_n\}$ is an exhaustion of X by finite connected circled sets (as explained in Sect. 1.2). Let s be a superharmonic function on X. If h_n is the Dirichlet solution on E_n with boundary value s on ∂E_n, then h_n is a decreasing sequence. If $\lim_{n\to\infty} h_n(x) = h(x)$, then either $h \equiv -\infty$ or h is the g.h.m. of s in X. In particular, if s is a potential then $\lim_{n\to\infty} [s(x) - h_n(x)] = s(x)$ for each $x \in X$.*

Proof. Suppose u is a harmonic function on E_n majorized by s. Then, by the Minimum Principle, $h_n - u \geq 0$ on E_n. Since h_{n+1} is harmonic and is majorized by s on $E_{n+1} \supset E_n$, we conclude that $h_n - h_{n+1} \geq 0$ on E_n. Since the limit of a decreasing sequence of harmonic functions is either $-\infty$ (with the convention $\infty - \infty = 0$) or finite everywhere, $h(x) = \lim_{n\to\infty} h_n(x)$ is either identically equal to $-\infty$ or is harmonic on X. In the latter case, if v is any harmonic function on X such that $v \leq s$, then $h_n \geq v$ on E_n and consequently, in the limit $h \geq v$ on X. That is, if h is harmonic on X, then it is the g.h.m. of s in X. The last assertion follows from the fact that if s is a potential, its g.h.m. is 0. □

Theorem 3.2.6. *(Potentials with point harmonic support) Let X be a hyperbolic network. Then, for any vertex e, there exists a unique potential $G_e(x)$ on X such that $\Delta G_e(x) = -\delta_e(x)$. Moreover, if a and b are two vertices in X, and if X has symmetric conductance, then $G_a(b) = G_b(a)$.*

Proof. Since X is hyperbolic, there exists a P-section $E = [e, e_1]$ determined by e. That is, there exists a bounded function $h \geq 0$ on E such that $h(e) = 0$, and $h(x) > 0$, $\Delta h(x) = 0$ for $x \in E \setminus \{e\}$. Extend h by 0 outside E so that as remarked in the proof of the above Theorem 3.2.4, $(-h)$ becomes a bounded superharmonic function on X. Let u be the g.h.m. of $(-h)$ so that $p = (-h) - u$ is a positive potential on X which is harmonic outside e. (Then, we say that p has *point harmonic support* at E.) Define now, $G_e(x) = \frac{p(x)}{(-\Delta)p(e)}$ on X. Then, $G_e(x)$ is a potential on X such that $\Delta G_e(x) = -\delta_e(x)$.

To prove the uniqueness, suppose v is a positive potential on X, such that $\Delta v(x) = -\delta_e(x)$. Then, $\Delta(G_e - v) = 0$ on X so that $G_e - v =$ a harmonic function H on X. Since $H \leq G_e$, we find $H \leq 0$; since $-H \leq v$, we find $-H \leq 0$. We conclude, $H \equiv 0$ and hence the uniqueness of the potential with point harmonic support.

To prove the symmetry, let us take an exhaustion $\{E_n\}$ of X by finite connected circled sets. Write $p(x) = G_a(x)$ and $q(x) = G_b(x)$. Let $u_n(x)$ be the Dirichlet solution on E_n with boundary value $p(x)$. Let $p_n(x) = p(x) - u_n(x)$ on E_n. Since $p(x)$ is a potential on X, by Lemma 3.2.5, $\lim_{n\to\infty} p_n(x) = p(x)$ for $x \in X$. Similarly, if $q_n(x) = q(x) - v_n(x)$ on E_n where $v_n(x)$ is the Dirichlet solution in E_n with boundary value $q(x)$, then $\lim_{n\to\infty} q_n(x) = q(x)$. Now, by the Green's Formula,

$$\sum_{\overset{0}{E_n}} [p_n \Delta q_n - q_n \Delta p_n] = -\sum_{\partial E_n} [p_n \frac{\partial q_n}{\partial n^-} - q_n \frac{\partial p_n}{\partial n^-}],$$

where for n large, $a, b \in \overset{\circ}{E}_n$. Since $\Delta q_n(x) = -\delta_b(x)$, $\Delta p_n(x) = -\delta_a(x)$, and $p_n = q_n = 0$ on ∂E_n, we find $-p_n(b) + q_n(a) = 0$. Allow now $n \to \infty$ to obtain $-p(b) + q(a) = 0$. That is, $G_a(b) = G_b(a)$. □

Note. The uniquely determined potential $G_e(x)$ with point harmonic support at e is bounded on X. (Recall that if s is a superharmonic function on X and if A is a subset of X such that $\Delta s(x) = 0$ for each x in $X \setminus A$, then s is said to have harmonic support in A.)

Corollary 3.2.7. *Let h be a harmonic function defined outside a finite set in a hyperbolic network. Then there exist on X, potentials p_1, p_2 with finite harmonic support and a unique harmonic function H, such that $h = p_1 - p_2 + H$ outside a finite set. The harmonic function H on X is bounded if and only if h is bounded.*

Proof. By hypothesis, we can assume that $\Delta h(x) = 0$ at every vertex $x \in X \setminus A$, where A is a finite set in X, and h is defined on ∂A. Extend h on $\overset{0}{A}$, by taking the Dirichlet solution with boundary values h on ∂A. Define $H(x) = h(x) + \sum_{z\in\partial A} [\Delta h(z)] G_z(x)$ on X. Note that $\Delta H(z) = \Delta h(z) + [\Delta h(z)](-1) = 0$ if $z \in \partial A$, and $\Delta H(x) = 0$ if $x \notin \partial A$. That is, $\Delta H \equiv 0$ on X, that is H is harmonic on X. Write

$$\sum_{z\in\partial A} [\Delta h(z)] G_z(x) = \sum_{z\in\partial A} [\Delta h(z)]^+ G_z(x) - \sum_{z\in\partial A} [\Delta h(z)]^- G_z(x)$$
$$= p_2(x) - p_1(x),$$

where p_1 and p_2 are two potentials with finite harmonic support in ∂A. Then, $h = p_1 - p_2 + H$ outside the finite set A.

To prove the uniqueness of the harmonic function H, let $h = q_1 - q_2 + H_1$ be another such representation outside some finite set. Then, $p_1 + q_2 + H = q_1 + p_2 + H_1$ outside a finite set. Note that if s is superharmonic on X and u is subharmonic on X such that $s \geq u$ outside a finite set, then by the Minimum Principle $s \geq u$ on X. Consequently, $H_1 - H \leq p_1 + q_2$ outside a finite set implies that the potential $p_1 + q_2$ majorizes the harmonic function $H_1 - H$ on X. Hence, $H_1 - H \leq 0$ on X; similarly, $H - H_1 \leq 0$. This proves that the harmonic function H is uniquely determined by h.

The remark about the boundedness of H comes from the construction of h, since $G_z(x)$ is bounded on X for any fixed $z \in X$. □

Corollary 3.2.8. *Let s be a non-negative superharmonic function on a hyperbolic network X. Let A be a finite set of vertices in X. Then, the balayage function R_s^A is a potential on X.*

Proof. Let $p(x) = \sum\limits_{a \in A} s(a) \frac{G_a(x)}{G_a(a)}$. Since A is a finite set, $p(x)$ is a potential on X; also, $p(x) \geq s(x)$ if $x \in A$. Hence, by Theorem 3.1.10, R_s^A is a potential on X. □

Theorem 3.2.9. *(Green's function on a set) Let F be an arbitrary set in a hyperbolic network. Then, for any $a \in \overset{0}{F}$, there exists a unique potential $G_a^F(x)$ on F such that $\Delta G_a^F(x) = -\delta_a(x)$ for $x \in \overset{0}{F}$, $G_a^F(z) = 0$ for $z \in \partial F$ and $G_a^F(x) \leq G_a(x)$ if $x \in F$.*

Proof. Take the Green potential $G_a(x)$ with point harmonic support in X (Theorem 3.2.6). Then, there exists (Theorem 3.1.7) a non-negative harmonic function h on F such that $h(z) = G_a(z)$ if $z \in \partial F$ and $0 \leq h(x) \leq G_a(x)$ on F. Define $G_a^F(x) = G_a(x) - h(x)$ if $x \in F$. Then, $G_a^F(x)$ has the stated properties. The uniqueness of $G_a^F(x)$ follows Corollary 3.1.4. □

3.2.2 Positive Harmonic Functions on a Network

If X is a parabolic network, then we know that any positive harmonic function on X is a constant. But in the case of a hyperbolic network, it is possible that there are non-constant positive harmonic functions also on X. To classify hyperbolic networks depending on the existence of positive harmonic functions on X, we introduce the following definition of *harmonic dimension of a hyperbolic network X*. (Heins [48] has introduced the notion of harmonic dimension of the point at infinity in the context of a parabolic Riemann surface.) A similar development in the case of axiomatic potential theory with positive potentials is given in [6].

Definition 3.2.2. Let e be a fixed vertex on a hyperbolic network X. Then, the harmonic dimension of X is the cardinality of the set of positive harmonic functions h on X such that $h(e) = 1$.

Note that the harmonic dimension of X is independent of the choice of the vertex e. For, if $\Im(e)$ is the family of positive harmonic functions h on X such that $h(e) = 1$ and $\Im(e_1)$ is the family of positive harmonic functions h_1 on X such that $h_1(e_1) = 1$, then the map $h(x) \rightarrow h_1(x) = \frac{h(x)}{h(e_1)}$ establishes a bijection between $\Im(e)$ and $\Im(e_1)$.

To determine the harmonic dimension of X, the sections determined by the vertex e are useful.

Theorem 3.2.10. *In a hyperbolic network, let $H_0^+(e)$ be the set of non-negative functions h on X such that $h(e) = 0$ and $\Delta h(x) = 0$ for each $x \neq e$. Let H^+ denote the set of non-negative harmonic functions on X. Then, there is a bijection T :*

$H_0^+(e) \to H^+$ such that if $h_1, h_2 \in H_0^+(e)$ and c_1, c_2 are non-negative constants, then $T(c_1h_1 + c_2h_2) = c_1T(h_1) + c_2T(h_2)$.

Proof. Let $h \in H_0^+(e)$. Since X is hyperbolic, $h = p_1 - p_2 + H$ outside a finite set, where H is a uniquely determined harmonic function on X (Corollary 3.2.7). Since $h \geq 0$, $-H \leq p_1 - p_2 \leq p_1$ outside a finite set, which implies that $-H \leq p_1$ on X. Hence, $H \geq 0$. Write $T(h) = H$, so that $|h - T(h)| \leq p = p_1 + p_2$ which is a potential. Suppose $u \in H^+$ such that $|h - u| \leq q$ where q is a potential in X. Then, $|u - T(h)| \leq p + q$ on X so that $u = T(h)$.

Thus, we can meaningfully define a map $T : H_0^+(e) \to H^+$ such that for $h \in H_0^+(e)$, $T(h)$ is the unique function in H^+ such that $|h - T(h)| \leq p$, where p is a potential in X. (Remark that $T(h)$ is the least harmonic majorant of the subharmonic function h on X. For, if $h = p_1 - p_2 + H$ outside a finite set, then $h - H = p_1 - p_2 \leq p_1$ outside a finite set so that $h - H \leq 0$ on X. Again, if v is a harmonic function on X such that $h \leq v$, then $H - v \leq p_2 - p_1 \leq p_2$, so that $H - v \leq 0$. Hence $H = T(h)$ is the least harmonic majorant of h on X. Consequently, if h is bounded then $T(h)$ is bounded.)

If $h_1, h_2 \in H_0^+(e)$, then $|h_1 - T(h_1)| \leq p_1$ and $|h_2 - T(h_2)| \leq p_2$ where p_1, p_2 are potentials on X. Consequently, for non-negative constants c_1, c_2 we have

$$|c_1h_1 + c_2h_2 - [c_1T(h_1) + c_2T(h_2)]| \leq c_1p_1 + c_2p_2.$$

Since $c_1p_1 + c_2p_2$ is a potential on X, $c_1T(h_1) + c_2T(h_2) = T(c_1h_1 + c_2h_2)$.

Now, T is a one-one map. For, if $T(h_1) = T(h_2)$, then the inequalities $|h_1 - T(h_1)| \leq q_1$ and $|h_2 - T(h_2)| \leq q_2$ outside a finite set would imply that $|h_1 - h_2| \leq q_1 + q_2$ outside a finite set. Now, $\Delta(h_1 - h_2) = 0$ at every vertex in $X \setminus \{e\}$, so that $\Delta(|h_1 - h_2|) \geq 0$ at every vertex in $X \setminus \{e\}$; since $|h_1(e) - h_2(e)| = 0$ also, it is immediate that $|h_1 - h_2|$ is subharmonic on X and also is majorized by the potential $q_1 + q_2$ outside a finite set. Hence, $|h_1 - h_2| \equiv 0$ on X, that is $h_1 = h_2$.

Finally to show that the map T is onto, take $u \in H^+$. Write $v = Su = u - R_u^e$. Recall that the balayage function R_u^e is a potential (Corollary 3.2.8) on X. Then, $v \geq 0$ on X, $v(e) = 0$, and $\Delta v(x) = 0$ when $x \neq e$. That is, $v \in H_0^+(e)$. Let $T(v) = f \in H^+$. Then, there exists a potential q on X such that $|v - f| \leq q$ on X. That is, $|(u - R_u^e) - f| \leq q$ on X, so that $|u - f| \leq q + R_u^e$. Now, $|u - f|$ is subharmonic on X and $q + R_u^e$ is a potential on X, so that $|u - f| \leq 0$. Consequently, $T(v) = u$. Hence, $T : H_0^+(e) \to H^+$ is an onto map. □

Note that if $h_1, h_2 \in H_0^+(e)$, then $h_2 = \lambda h_1$ for some constant $\lambda > 0$, if and only if $T(h_2) = \lambda T(h_1)$. Consequently, if we introduce an equivalence relation $\sim$ in $H_0^+(e)$ by saying that $h \sim h'$ if and only if $h' = \lambda h$ for some $\lambda > 0$, then the number of non-zero equivalence classes in $H_0^+(e)$ is the harmonic dimension of X.

Proposition 3.2.11. *In a hyperbolic network X, if a vertex e determines more than one infinite section, then there exists a non-constant harmonic function $h > 0$ on X.*

Proof. In any network, an infinite section $E = [e, e_1]$ determined by a vertex e introduces a function $h \in H_0^+(e)$. For, there is a function $h \geq 0$ on E such that $h(e) = 0$ and $h(x) > 0$ and $\Delta h(x) = 0$ if $x \in E \backslash e$. Extend h by 0 outside E to define a function $h \in H_0^+(e)$. This h can be chosen as a bounded function if E is a P-section and h is unbounded if E is an S-section.

Here, since X is hyperbolic, any vertex should determine at least one P-section. If the second infinite section also is a P-section, then there are two bounded non-proportional harmonic functions in $H_0^+(e)$; and they correspond to two bounded non-proportional harmonic functions in H^+ by Theorem 3.2.10. Hence, at least one of them is a bounded non-constant harmonic function on X.

On the other hand, if the second infinite section is an S-section, then there exists an unbounded harmonic function in $H_0^+(e)$ and hence there is an unbounded harmonic function in H^+. □

3.2.3 Integral Representation of Positive Harmonic Functions

Any positive harmonic function in the ball $B(0, R)$ in $\Re^n$, $n \geq 2$, has a Poisson-Stieltjes integral representation $\int\limits_S R^{n-2} \frac{R^2 - |x|^2}{|x-y|^n} d\mu(y)$ where μ is a positive regular Borel measure on the boundary S. In 1941, Martin obtained this representation (known now as the *Martin Representation*) for positive harmonic functions defined on any bounded domain Ω in $\Re^n$. To achieve that, Martin introduced a new topology on Ω such that there exists a compact space $\overset{\wedge}{\Omega}$ in which Ω is dense open. The boundary $\overset{\wedge}{\Omega} \backslash \Omega$ (known as the *Martin boundary* of Ω) is generally different from the Euclidean boundary $\partial\Omega$ of Ω. It is a fascinating problem to look for domains Ω for which the Martin boundary can be identified with the Euclidean boundary. When Ω is a ball, the boundaries are the same (see Brelot [29]); for some other domains for which the two boundaries can be identified, we can cite Lipschitz domains which include all bounded domains with smooth boundaries [49], Nontangentially accessible domains [50] and Uniform domains [4]; if Ω is the half-space in the Euclidean space $\Re^n$, then the Martin compactification $\overset{\wedge}{\Omega}$ is homeomorphic to $\overline{\Omega} \cup \{\infty\}$. Later in 1956, Choquet obtained the Martin Representation in an abstract setting dealing with extremal elements and barycentres. In this paragraph, we use the Choquet integral representation theory [39, pp. 299–309] in the context of positive harmonic functions defined on a hyperbolic network X. Let H_1^+ denote the class of positive harmonic functions on X taking the value 1 at a fixed vertex e.

Proposition 3.2.12. *Suppose $\{h_n\}$ is a sequence in H_1^+ converging to h at each vertex in X. Then $h \in H_1^+$.*

Proof. It is enough to prove that h is finite at every vertex in X. Recall (Sect. 3.1) the Harnack property in X: Given two vertices a and b in X, there are positive numbers α and β, depending only on a and b, such that $\alpha s(b) \leq s(a) \leq \beta s(b)$ for any positive superharmonic function s on X.

Let y be an arbitrary vertex in X. Then $\alpha h_n(e) \leq h_n(y) \leq \beta h_n(e)$. Hence $\{h_n(y)\}$ is bounded and since $h_n \to h$, we see that $h(y)$ is finite. The vertex y being arbitrary, we conclude that h is real-valued and further as the limit of a sequence of harmonic functions, h is harmonic on X and $h(e) = 1$. That is, $h \in H_1^+$. □

Proposition 3.2.13. *Suppose $\{h_n\}$ is a sequence of harmonic functions in H_1^+. Then there exists a subsequence of $\{h_n\}$ converging to a function h in H_1^+.*

Proof. We have just remarked that for any y in X, $\alpha \leq h_n(y) \leq \beta$. Since $\{h_n(y)\}$ is bounded, then there is a subsequence $\left\{h_n^{'}\right\}$ of $\{h_n\}$ such that $\lim_{n\to\infty} h_n^{'}(y)$ is finite. Let z be another vertex in X. Then, as before we can extract a subsequence $\left\{h_n^{''}\right\}$ from $\left\{h_n^{'}\right\}$ such that $\lim_{n\to\infty} h_n^{''}(z)$ is finite; clearly, $\lim_{n\to\infty} h_n^{''}(y)$ also is finite. Since X has only a countable number of vertices, the above method allows us to extract a subsequence from the original sequence $\{h_n\}$, converging to a real-valued function h on X. Then, by the above Proposition 3.2.12, $h \in H_1^+$. □

Remark that H^+ is a convex cone, that is if $u, v \in H^+$, then $\alpha u + \beta v \in H^+$, where α, β are any non-negative numbers. Further, $\inf(u, v)$ is a non-negative superharmonic function on X. Let h be the greatest harmonic function on X (see Theorem 2.4.10) such that $0 \leq h \leq \inf(u, v)$. Similarly, we can find the smallest harmonic function w in H^+ such that $\sup(u, v) \leq w \leq u + v$. Thus, H^+ is a lattice for the natural order. If $f, g \in H^+$, define $\|f - g\| = \sup_{x\in X} \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|}$. Let us write $E = H^+ - H^+$. If $f, g \in E$, define $\|f - g\| = \sup_{x\in X} \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|}$. Then, E provided with the vertexwise convergence is locally convex and metrisable. In E, H^+ is a convex cone which is a lattice for the natural order, and by Proposition 3.2.13, $H_1^+$is a compact base.

A *base* for a convex cone C with vertex 0 in a locally convex space E is the convex subset $B = \pi \cap C$ where π is a hyperplane not containing 0 but intersecting all the generators of C. An element $u \in C$ is said to be *extremal* if whenever u has a representation of the form $u = \lambda u_1 + (1 - \lambda)u_2$ where $u_1, u_2 \in C$ and $0 \leq \lambda \leq 1$, then u is either u_1 or u_2. That is, u is an end point of any segment in C that contains u. An element $v \in C$ is said to be *minimal* if every $w \in C$, for which $v - w \in C$, is of the form $w = \alpha v$ for some $\alpha \geq 0$. Suppose u is extremal; then it is minimal. For, if $w \in C$ and $u - w \in C$, then $u = w + (u - w)$. It is possible $w = 0$ or $w = u$. Otherwise, $u = \alpha\left(\frac{w}{\alpha}\right) + (1 - \alpha)\left(\frac{u-w}{1-\alpha}\right)$ for $0 < \alpha < 1$. Since u is extremal, then $u = \frac{w}{\alpha}$ or $u = \frac{u-w}{1-\alpha}$ which implies that $w = \alpha u$. We shall quote now the famous Choquet Theorem [34].

Theorem 3.2.14. *(Choquet) Let E be a real linear Hausdorff space that is locally convex. Let C be a convex cone in E with compact base B. Let Ξ be the set of extremal points in B. Then,*

i. *If B is metrisable, then every $x \in B$ is the barycentre of a unitary measure μ with support in Ξ. That is, there exists a measure $\mu \geq 0$ with $\|\mu\| = 1$ and $\mu(\Xi) = 1$ such that if l is any continuous linear functional, then $l(x) = \int l(s)d\mu(s)$.*
ii. *If the cone C is a lattice for the order it defines, then the measure μ defined above is unique.*

Let us use the theorem in the context of hyperbolic networks. H^+ is the convex cone C, $E = H^+ - H^+$, the base $B = H_1^+$. Let Λ_1 be the set of minimal points in H_1^+. Recall that every extremal point is minimal, H^+ is a lattice and H_1^+ is compact. Consequently,

Theorem 3.2.15. *Let X be a hyperbolic network. Then, there exists a unique unitary measure $\mu \geq 0$ with support in Λ_1 such that for any $h \in H_1^+$ we have $h(x) = \int\limits_{u \in \Lambda_1} u(x)d\mu(u)$, for $x \in X$.*

Corollary 3.2.16. *Let v be any positive harmonic function on a hyperbolic network X. Then, there exists a unique positive measure ν with support in Λ_1 such that $v(x) = \int\limits_{u \in \Lambda_1} u(x)d\nu(u)$ for every $x \in X$.*

Proof. Since $\frac{v(x)}{v(e)} \in H_1^+$, by the above theorem there exists a unique positive measure μ with support in Λ_1 such that $\frac{v(x)}{v(e)} = \int\limits_{u \in \Lambda_1} u(x)d\mu(u)$. Write $d\nu(u) = v(e)d\mu(u)$. Then, $v(x) = \int\limits_{u \in \Lambda_1} u(x)d\nu(u)$ for any $x \in X$. □

3.3 Hyperbolic Networks

The fact that the Dirichlet problem in a generalised form has a solution (Theorem 3.1.7) is of immense significance in an infinite network. For, some of the potential-theoretically important results (useful in infinite electrical networks and random walks) like Balayage, Poisson kernel, Domination principle, Condenser principle, Green kernel on a subset, Capacitary functions, Dirichlet-Poisson solution etc. can be obtained as solutions to appropriately-posed Dirichlet problems. In this section, we assume that there exist positive potentials in the network X. Hence, by Theorem 3.2.6, for any vertex e in X, there exists a unique potential $G_e(x)$ on X such that $\Delta G_e(x) = -\delta_e(x)$, for all $x \in X$.

Theorem 3.3.1. *A superharmonic function p on a hyperbolic network X is a potential if and only if $p(x) = \sum\limits_{y \in X} (-\Delta)p(y)G_y(x)$, for all $x \in X$.*

Proof. Suppose p is a superharmonic function on X and the above equality holds. Since each term $(-\Delta)p(y)G_y(x)$ is a potential in X, if $\sum\limits_{y \in X} (-\Delta)p(y)G_y(x)$ is convergent for each x, then the sum is a potential.

Conversely, let p be a potential on X. Let $\{E_n\}$ be an exhaustion of X by finite connected circled sets. Let $p_n(x) = \sum_{y \in E_n} (-\Delta)p(y)G_y(x)$. Then, p_n is a potential on X. Let $s_n(x) = p(x) - p_n(x)$ on X. Then, for $x \in E_n$, $\Delta p_n(x) = \Delta p(x)$ so that $\Delta s_n(x) = 0$ on E_n; if $x \notin E_n$, $\Delta p_n(x) = 0$ while $\Delta p(x) \leq 0$ so that $\Delta s_n(x) \leq 0$ if $x \in X \backslash E_n$. Consequently, $\Delta s_n \leq 0$ on X. That is, s_n is superharmonic on X and $p_n + s_n = p > 0$, so that $-s_n < p_n$ on X. Now, p_n is a potential on X majorizing the subharmonic function $-s_n$. Hence $-s_n \leq 0$. Consequently, $p = p_n + s_n \geq p_n$ for any n. Allow $n \to \infty$, to conclude that $q(x) = \sum_{y \in X} (-\Delta)p(y)G_y(x) \leq p(x)$.

Since $q(x)$ is the limit of an increasing sequence of potentials p_n, then $q(x)$ is a non-negative superharmonic function which is also majorized by the potential $p(x)$. Hence, $q(x)$ is a potential. Let $h(x) = p(x) - q(x)$. Then,

$$\Delta h(x) = \Delta p(x) - \Delta q(x) = \Delta p(x) - [(-\Delta)p(x) \times (-1)] = 0,$$

so that h is harmonic on X. Consequently, $p(x) = q(x) + h(x)$ implies, by the uniqueness of decomposition of non-negative superharmonic functions, that $h \equiv 0$. Hence, $p(x) = \sum_{y \in X} (-\Delta)p(y)G_y(x)$, for all $x \in X$. □

The above representation of a positive potential on X is taken as the definition of a potential in a tree T in Cartier [31, Sects. 2.2 and 2.3]. There, by starting with the notion of the kernel associated to a collection of paths, the Green kernel $G(x, y)$ is defined. Then, for any function $f \geq 0$ on T, $Gf(x) = \sum_y G(x, y)f(y)$ is either always ∞ or always finite on T such that $\Delta Gf = -f$; in the latter case, $Gf(x)$ is termed as the potential of f. A different approach to this development of potential theory using the operators runs as follows (when we know that there are positive potentials on X).

Introduce an operator A by defining, for a function f on a network X,

$$Af(x) = \sum_{y \sim x} \frac{t(x, y)}{t(x)} f(y), \text{for } x \in X. \text{ Then,}$$

$$\begin{aligned}\Delta f(x) &= \sum_{y \sim x} t(x, y)[f(y) - f(x)] \\ &= t(x) \left[\sum_{y \sim x} \frac{t(x, y)}{t(x)} f(y) - f(x) \right] \\ &= t(x)(A - I)f(x).\end{aligned}$$

Hence, a function u on X is superharmonic if and only if $u(x) \geq Au(x)$. Note also that A is linear; if $f \geq g$, then $Af \geq Ag$; and if $f_n \to f$, then $Af_n \to Af$. Writing $A^{n+1}f = A[A^n f]$ for $n \geq 1$, define the operator K by $Kf(x) = \sum_{n=0}^{\infty} A^n f(x)$.

Lemma 3.3.2. *Let s be a positive superharmonic function on X. Then, the g.h.m. h of s is given by $h(x) = \lim_{n\to\infty} A^n s(x)$.*

Proof. Since s is superharmonic, $As \leq s$. From this, we deduce that $A^n s \leq A^{n-1}s \leq s, n \geq 2$. Hence, $h(x) = \lim_{n\to\infty} A^n s(x)$ exists. Since $A^n s \geq A(A^n s)$, taking limits, we find that $h \geq Ah$; since $A(A^{n-1}s) \geq A^{n+1}s$, again taking limits we see that $Ah \geq h$. Hence, h is harmonic on X and $h \leq s$.

Suppose now u is any non-negative harmonic function such that $u \leq s$. Then, $u = Au \leq As$. Proceeding similarly, for any n, we see that $u \leq A^n s$. Taking limits, when $n \to \infty$, we see that $u \leq h$. Thus, $h = \lim_{n\to\infty} A^n s$ is the g.h.m. of s. □

Theorem 3.3.3. *A function u on X is a potential if and only if $u = Kf$ for some $f \geq 0$.*

Proof. Suppose $u = Kf = \sum_{n=0}^{\infty} A^n f$ for some $f \geq 0$. Then, $u \geq 0$ and

$$Au = AKf = Kf - f = u - f \leq u,$$

so that u is a non-negative superharmonic function on X. Let h be the g.h.m. of u on X. Then, $0 \leq h \leq u$ and $h = Ah \leq Au$; continuing this argument, we arrive at $h \leq A^m u$ for any positive integer m. Since $u(x) = \sum_{n=0}^{\infty} A^n f(x)$ converges for every vertex $x \in X$, we note that $A^m u(x) = \sum_{n=m}^{\infty} A^n f(x)$ so that $\lim_{m\to\infty} A^m u(x) = 0$, and hence $h(x) = 0$. This leads to the conclusion $h \equiv 0$ and hence u is a potential on X.

Conversely, suppose u is a potential on X. That is, the g.h.m. of u is 0. Now by Lemma 3.3.2, we know that the g.h.m. of u is $\lim_{n\to\infty} A^n u(x)$. Hence, for every $x \in X$, $\lim_{n\to\infty} A^n u(x) = 0$. Consequently,

$$\begin{aligned} K(u - Au) &= \sum_{n=0}^{\infty} A^n (u - Au) \\ &= u - \lim_{n\to\infty} A^n u \\ &= u. \end{aligned}$$

Thus, the given potential $u = Kf$ where $f = u - Au \geq 0$. □

Corollary 3.3.4. *Let s be a positive superharmonic function on X. Then, s is the unique sum of a potential p and a non-negative harmonic function h on X.*

Proof. Since s is superharmonic, $v = s - As \geq 0$ on X. Now, $s = v + As$ implies that $As = Av + A^2 s$ and hence, $s = v + Av + A^2 s$. Similar arguments give the equality $s = v + Av + A^2 v + ... + A^n v + A^{n+1} s \geq v + Av + ... + A^n v$. Allow $n \to \infty$ to see that $s \geq Kv$ on X. Let $h = s - Kv \geq 0$. Then, since $K = I + KA = I + AK$, we find

$$Ah = As - AKv = As - (Kv - v) = (As + v) - Kv = s - Kv = h,$$

so that h is a non-negative harmonic function on X. Note $Kv = p$ is a potential by the above theorem. Thus, we have the Riesz decomposition for the positive superharmonic function s as $s = p + h$.

Suppose s is the sum of another potential (which should be of the form Kf by the above theorem) and a non-negative harmonic function H on X. Now, $s = Kf + H$ implies that $As = AKf + AH = AKf + H$, so that $s - As = Kf - AKf = f$; hence, $v = s - As = f$. Consequently, $h = H$. Thus, the decomposition of s as the sum of a potential and a non-negative harmonic function is unique. □

Proposition 3.3.5. *Let Kf be a potential for some $f \geq 0$ on X. Let $E = \{x : f(x) > 0\}$. Suppose $s \geq 0$ is a superharmonic function on X such that $s \geq Kf$ on the set E. Then $s \geq Kf$ on X.*

Proof. Let $p = \inf\{s, Kf\}$ on X. Since p is majorized by the potential Kf, p itself is a potential and hence $p = Kg$ for some $g \geq 0$. Note that $Kg \leq Kf$ on X and $Kg = Kf$ on E. Hence, if $x \in E$, then $g(x) = Kg(x) - AKg(x) \geq Kf(x) - AKf(x) = f(x)$. If $y \notin E$, then $f(y) = 0 \leq g(y)$. Hence $f \leq g$ on X, so that $p = Kg \geq Kf$ also on X. Consequently, $p = Kg = Kf$ on X, and hence $s \geq Kf$ on X. □

We shall give a variation of the above result in the following theorem, without involving the operator K. In $\Re^3$, the Maria-Frostman maximum principle [27, p.118] states that if v is the potential $v(x) = \int |x - y|^{-1} d\mu(y)$ where $\mu \geq 0$ is a Radon measure with support in a closed set A, then $v(x) \leq \sup_{y \in A} v(y)$ for all $x \in \Re^3$. Recall that in a network, hyperbolic or parabolic, if s is a superharmonic function, then the set $E = \{x : \Delta s(x) < 0\}$ is called *the harmonic support* of s.

Theorem 3.3.6. *(Domination Principle) Let p be a potential on X. Suppose s is non-negative superharmonic on X such that $s \geq p$ on the harmonic support of p. Then, $s \geq p$ on X.*

Proof. Let $q = \inf(s, p)$. Then, q is a potential on X such that $q \leq p$ on X and $q = p$ on E, the harmonic support of p. Let $\varphi = p - q$ on X. Then $\Delta\varphi \geq 0$ on $X \setminus E$, $\varphi \geq 0$ on X, and $\varphi = 0$ on E, so that $\Delta\varphi \geq 0$ on X. Thus, φ is a subharmonic function on X, majorized by the potential p on X, and consequently, $\varphi \equiv 0$. This proves that $p = q = \inf(s, p)$ on X, so that $s \geq p$ on X. □

Corollary 3.3.7. *Let $G_e(x)$ be the potential with point harmonic support e in X. Then $G_e(x) \leq G_e(e)$ for all $x \in X$. In fact, $G_e(x) = G_e(e) R_1^e(x)$ on X.*

Proof. $G_e(x)$ is a potential majorized by $G_e(e)$ on its harmonic support e. Consequently, $G_e(x) \leq G_e(e)$ for all $x \in X$. Let now $q(x) = \frac{G_e(x)}{G_e(e)}$ Then, $q(x)$ and $R_1^e(x)$ are potentials with the same harmonic support e and $q(e) = R_1^e(e) = 1$,(see Theorem 3.1.10 and Corollary 3.1.9). Hence, by the Domination Principle, $q(x) \equiv R_1^e(x)$. □

Remark 3.3.1. 1. The form of the Green potential $G_e(x)$: Let $\omega = [e, e_1]$ be an S-section determined by e. Suppose $G_e(e_1) = G_e(e)$. Then by the Maximum Principle, since $e_1 \in \overset{0}{\omega}$, we conclude that $G_e(x) = G_e(e)$ in ω. Otherwise, $G_e(e_1) < G_e(e)$. For $x \in \omega$, let

$$h(x) = \frac{G_e(e) - G_e(x)}{G_e(e) - G_e(e_1)}.$$

Then $h(x) \geq 0$ is harmonic and bounded on ω (Corollary 3.3.7). Then $h \equiv 0$ or $h > 0$ on $\overset{0}{\omega}$. If the latter is true, then $h(e) = 0$ and $h(e_1) = 1$ so that ω should be a P-section, a contradiction. Hence $h \equiv 0$, that is $G_e(x) = G_e(e)$ for every $x \in \omega$.

Thus, $G_e(x)$ is the constant $G_e(e)$ in each of the S-sections determined by e. This result accords with a result in the special case of a star-shaped tree, given in Cartier [31, pp. 259–260].

2. In Gowrisankaran and Singman [47, Theorem 4.3], the Domination Principle (Theorem 3.3.6) is proved in the context of an infinite tree T, assuming that the Green kernel $G(s,t)$ is such that $0 < G(s,t) < \infty$ for all s, t in T and the limit of $G(s,t)$ is 0 when $|s|$ goes to infinity for some (equivalently for every) vertex t. The latter condition is not satisfied in every tree T, as shown in [47, Example 2.1]. An assumption sufficient for the latter condition to be satisfied is: There exists a constant δ such that $0 < \delta < \frac{1}{2}$ and for all $s_1, s_2 \in T$ with $s_1 \sim s_2$, we should have $\delta \leq t(s_1, s_2) \leq \frac{1}{2} - \delta$.
3. *Bôcher's Theorem*: In $\Re^n, n \geq 2$, let B be the unit ball $|x| < 1$. Suppose u is positive harmonic on $B \setminus \{0\}$. Then there exist a harmonic function v on B and a constant $b \geq 0$ such that in $B \setminus \{0\}$,
(a) $u(x) = b \log \frac{1}{|x|} + v(x)$ if $n = 2$.
(b) $u(x) = b\,|x|^{2-n} + v(x)$ if $n \geq 3$.
A proof of Bôcher's theorem can be found in Axler et al. [13, pp.50–54]. This property in the axiomatic case is stated [28, p. 40] as the *axiom of proportionality*: *If p and q are two positive potentials in a harmonic space with the same point harmonic support, then $p = \lambda q$ for some positive constant λ.*

Thus, the Bôcher's theorem asserts that the logarithmic potential $\log \frac{1}{|x|}$ in $\Re^2$ and the potential $|x|^{2-n}$ in $\Re^n, n \geq 3$, are unique (up to a multiplicative constant) functions in the study of positive harmonic functions with point singularity. In the context of a network X, the above Theorem 3.3.6 stating the Domination Principle proves also an enlarged version of the analogue of Bôcher's Theorem: *Let p and q be two potentials on a hyperbolic network X, having the same harmonic support A. If $p = q$ on A, then $p = q$ on X.*

Theorem 3.3.8. *Let A be a proper subset of a hyperbolic network. Let f be a real-valued function defined on A. Suppose there exists a potential p on X such that $|f| \leq p$ on A. Then, there exists a unique function h on X such that $h = f$ on A, $\Delta h = 0$ at every vertex in $X \setminus A$ and $|h|$ is majorized by a potential on X.*

Proof. In Theorem 3.1.7, take $E = X\backslash A$, $F = V(E)$, $u = p$ and $v = -p$. Then, there exists a function h on $V(E)$ such that $\Delta h(x) = 0$ if $x \in E$, $h = f$ on $V(E)\backslash E$, and $-p \leq h \leq p$ on $V(E)$. Let us assume that h is extended by f outside $V(E)$. Thus, h is a function on X such that $\Delta h(x) = 0$ if $x \in X\backslash A$, $h = f$ on A and $|h| \leq p$ on X. Note that if $0 \leq f \leq p$ on A, then $h \geq 0$ on X.

To prove the uniqueness, suppose u is another function on X such that $\Delta u(x) = 0$ if $x \in X\backslash A$, $u = f$ on A and $|u| \leq q$ where q is a potential on X. Let $\varphi = h - u$ on X. Then, $\Delta\varphi(x) = 0$ at every vertex in $X\backslash A$, $\varphi = 0$ on A and $|\varphi| \leq p + q$ on X. That means that $|\varphi|$ is a subharmonic function on X, majorized by the potential $p + q$ on X. Hence, $\varphi \equiv 0$. □

Corollary 3.3.9. *Let f be a real-valued function defined on a finite subset F of a hyperbolic network X. Then, there exists a unique function h on X dominated by a potential such that $h = f$ on F and $\Delta h(x) = 0$ if $x \in X\backslash F$. The function f can also be represented as $f = p_1 - p_2$ on F, where p_1, p_2 are potentials in X with harmonic support in F.*

Proof. Let $q(x)$ be a positive potential on X. Since f is a finite set, there is some $M > 0$ such that $|f| \leq M$ on f. Let $p(x) = \frac{M}{\inf_{y\in F} q(y)} q(x)$ on X. Then, $p(x)$ is a potential such that $|f(x)| \leq p(x)$ if $x \in F$. Then the first part of the corollary follows from the above theorem.

Now, let $p_1(x) = \sum_{a\in F} [(-\Delta h(a)]^+ G_a(x)$ and $p_2(x) = \sum_{a\in F} [(-\Delta h(a)]^- G_a(x)$. Then, p_1 and p_2 are potentials on X such that $\Delta(p_1 - p_2) = \Delta h$ on X and p_1, p_2 have harmonic support in f. Hence, $h = p_1 - p_2 + u$ on X, where u is harmonic on X. Since h is dominated by a potential on X, that is $|h| \leq v$, where v is a potential on X, $|u| \leq v + p_1 + p_2$ on X which implies that $u \equiv 0$. Consequently, $f = p_1 - p_2$ on X. □

Remark 3.3.2. 1. Like sets of measure 0 in measure theory and nowhere dense sets in topology, in the Euclidean space $\Re^n, n \geq 2$, *a polar set* is a negligible set in the potential-theoretic sense. The above Corollary 3.3.9 is a discrete analogue of a generalised version of a result in the study of capacities and polar sets in the classical potential theory, namely: Let K be a compact non-polar set in $\Re^3$. (A set k in a domain ω in $\Re^n, n \geq 2$, is said to be a *polar set* if for a superharmonic function $s > 0$ on ω, $k \subset \{x : s(x) = \infty\}$. Brelot in 1941 introduced the notion of polar sets to replace the sets of inner capacity 0, which were in vogue till then.) Then there exists a (Newton) potential $u > 0$ on $\Re^3$ such that $u = 1$ on K except possibly on a polar set and $\Delta u = 0$ on $\Re^3\backslash K$. This very useful potential is called the *capacitary potential* of K, see Brelot [27, pp.52–53] and Armitage and Gardiner [10, Sect. 5.4].

In fact, inspired by the notion of capacity in electrostatics, De La Vallée Poussin defined $Cap(K)$, the capacity of a compact set K in $\Re^3$ as the upper bound of the masses m distributed on K such that the associated potentials

$\int |x-y|^{-1} \, dm(y)$ are majorized by 1 on $\Re^3$. In $\Re^2$, we use the logarithmic kernel $-\log|x-y|$ (instead of the Newton kernel $|x-y|^{-1}$ as in $\Re^3$) to define the *logarithmic capacity* of a compact set K. Then, for an arbitrary set A in $\Re^3$ or $\Re^2$, the *inner capacity* $Cap_*(A)$ of A is defined as $Cap_*(A) = \sup_{\Im} Cap(K)$, where $\Im$ is the family of all compact sets K contained in A. This is similar to defining the inner measure of a set in measure theory. Soon the importance of sets of inner capacity 0 was recognized in the classical potential theory. The notion of inner capacity 0 is stronger than the notion of inner Lebesgue measure 0, in the sense that a set of inner capacity 0 has its inner Lebesgue measure 0 but not conversely. For example, the X-axis in $\Re^2$ is of measure 0 but its inner capacity is not 0.

G.C. Evans in 1936 characterized a compact set K of capacity 0 in $\Re^3$ by the existence of a non-negative measure μ on K such that $K = \left\{x : \int |x-y|^{-1} \, d\mu(y) = \infty\right\}$. In this context, knowing that potentials are non-negative superharmonic functions, M.Brelot introduced the *polar sets* to replace the sets of inner measure 0. Soon after, H.Cartan remarked that a set E is polar if and only if its *outer capacity* is 0.

Such a notion of polar sets in the context of potential theory on infinite networks or trees is irrelevant. For, if u is an extended real-valued function on a network or a tree X, such that $u(x) > -\infty$ for all x in X and $\Delta u(x) \leq 0$ at every vertex x in X (with the convention $\infty - \infty = 0$) and if u takes the value ∞ at one vertex in X, then $u \equiv \infty$. This fact is not surprising since a network has much in common with one-dimensional manifolds and on the real line $\Re$, considered as a harmonic space, every point is non-polar [27, p.47].

2. Write $(-\Delta)h(a) = \xi(a)$ in the above corollary. Then Corollary 3.3.9 can be stated as follows: *Let F be a finite subset in a hyperbolic network X. If f is a real-valued function on F, then there exists a function $\xi(x)$ with finite support in F such that $f(x) = \sum_{y \in X} \xi(y) G_y(x)$ for $x \in F$.*

In a finite network X with symmetric conductance, when a proper subset F is given, Bendito et al. 27, [18] construct a unique function $\nu^F \geq 0$ on X such that $\Delta \nu^F(x) = -1$ if and $\left\{x : \nu^F(x) \neq 0\right\} = F$, by solving *linear programming problems*. By calculating ν^F for appropriate subsets F, they obtain the Maximum Principle, the Dirichlet solution, the Condenser Principle etc. in a finite network with symmetric conductance. The theorem below introduces a function analogous to ν^F in the context of an infinite network (which according to our definition need not have symmetric conductance).

Theorem 3.3.10. *Let F be a finite set of vertices in an infinite network X (hyperbolic or parabolic). Let f be a function defined on F. Then there exists a unique function u on X such that $\Delta u(x) = -f(x)$ if $x \in F$, and $u(y) = 0$ if $y \in X \backslash F$. Further, $u \geq 0$ if $f \geq 0$, and if $f > 0$ on F, then $u > 0$ on F.*

Proof. Take $E = V(F) = \bigcup_{x \in F} V(x)$, which is the smallest set containing F and all the neighbours of the vertices in F. Remark that $F \subset \overset{0}{E}$. For every $a \in F$, by using the Dirichlet solution, construct a non-negative function $g_a(x)$ on E such that $g_a(x) = 0$ on $E \backslash F$ and $\Delta g_a(x) = -\delta_a(x)$ if $x \in F$; extend $g_a(x)$ by giving the value 0 on $X \backslash E$ (Theorem 2.4.15). Associated to the given function f on the finite set F, define two non-negative functions u^+ and u^- on X such that $u^+(x) = \sum_{a \in F} f^+(a) g_a(x)$ if $x \in F$ and $u^+(y) = 0$ if $y \in X \backslash F$; and $u^-(x) = \sum_{a \in F} f^-(a) g_a(x)$ if $x \in F$ and $u^-(y) = 0$ if $y \in X \backslash F$. If $u = u^+ - u^-$ on X, then $\Delta u(x) = -f(x)$ if $x \in F$ and $u(y) = 0$ if $y \in X \backslash F$. Also, if $f \geq 0$, then $u = u^+ \geq 0$. Finally, suppose $f > 0$ on F and $u(a) = 0$ for some $a \in F$. Then,

$$
\begin{aligned}
-f(a) = \Delta u(a) &= \sum_{x \sim a} t(a,x)[u(x) - u(a)] \\
&= \sum_{x \sim a} t(a,x) u(x) \geq 0,
\end{aligned}
$$

a contradiction. Consequently, if $f > 0$ on F, then $u > 0$ on F.

If v is a function on X such that $\Delta v(x) = 0$ for $x \in F$, and $v(y) = 0$ if $y \in X \backslash F$, then by the Maximum Principle $v \equiv 0$. This proves that the solution u is uniquely determined by the given conditions. □

Theorem 3.3.11. *(Dirichlet-Poisson solution) Let E be a finite set in a hyperbolic network. Let f and g be two functions defined on X. Then, there exists a unique function v on X such that $\Delta v = g$ on E and $v = f$ on $X \backslash E$.*

Proof. Let $F = V(E) = \bigcup_{x \in E} V(x)$. Since E and F are finite sets, by Theorem 3.1.7, there exists a unique function h on F such that $\Delta h(x) = 0$ if $x \in E$ and $h = f$ on $F \backslash E$. Assume that h is defined on X, by giving values of f outside F. Now, by Theorem 3.3.10, there exists a unique function u on X such that $\Delta u(x) = g(x)$ if $x \in E$ and $u(y) = 0$ if $y \in X \backslash E$. Write $v = u + h$ on X. Then, $\Delta v = g$ on E and $v = f$ on $X \backslash E$. The uniqueness of the function v follows from the Maximum Principle. □

Remark 3.3.3. The above results do not prove that given an arbitrary function f on an infinite network X, there exists a function u on X such that $\Delta u = f$ on X. However from [2 Theorem 6] we can see that if X is a hyperbolic network with the Green function $G(x,y)$ satisfying the condition $\sum_{a \in X} G(a,a)$ is finite and if f is a bounded function, then

$$
u(x) = -\sum_{y \in X} f(y) G_y(x) = \sum_{y \in X} f^-(y) G_y(x) - \sum_{y \in X} f^+(y) G_y(x)
$$

is a δ-superharmonic function (that is, a difference of two superharmonic functions) such that $\Delta u = f$ on X.

Also from the results in Sect. 5.1, Chap. 5, we can deduce that if X is an infinite tree without terminal vertices (whether X is hyperbolic or parabolic), then given an arbitrary real-valued function f on X, we can determine a δ-superharmonic function u (that is, u is a difference of two superharmonic functions) such that $\Delta u = f$ on X.

The following result shows that a positive superharmonic function on a hyperbolic network can be represented by two positive measures. This is an extension of the integral representation of positive harmonic functions given in Corollary 3.2.16, known as the *Riesz-Martin representation of positive superharmonic functions*. In the case of axiomatic potential theory, it is given in Brelot [28, pp.83–84].

Theorem 3.3.12. *Let s be a positive superharmonic function defined on a hyperbolic network X. Then, there exist two uniquely determined positive measures μ in X and ν with support in Λ_1 such that $s(x) = \int_{y \in X} G_y(x) d\mu(y) + \int_{u \in \Lambda_1} u(x) d\nu(u)$, for every x in X.*

Proof. We have already remarked that we can write $s = p + h$ as the unique sum of a potential and a non-negative harmonic function on X. Write $\mu(y) = -\Delta p(y)$. Then, $\mu \geq 0$ can be considered as a positive measure on X such that (Theorem 3.3.1) $p(x) = \sum_{y \in X} G_y(x)\mu(y) = \int_{y \in X} G_y(x) d\mu(y)$. As for the non-negative harmonic function h on X, we have an integral representation (Corollary 3.2.16) $h(x) = \int_{u \in \Lambda_1} u(x) d\nu(u)$ for any x in X. This gives the unique representation of the superharmonic function $s \geq 0$ by means of two uniquely determined positive measures μ and ν. □

3.4 Parabolic Networks

An infinite network X is referred to as *a parabolic network* if and only if there is no positive potential on X (Definition 3.1.2). Thus, an infinite network is parabolic if and only if any lower bounded function s on X is a constant if $\Delta s \leq 0$.

Example of a parabolic network: Let $X = \{z_1, z_2, x_1, x_2, x_3, ...\}$ together with an infinite set of terminal vertices $\{y_{21}, y_{22}, y_{31}, y_{32}, ...\}$, where y_{i1}, y_{i2} have x_i as their unique neighbour for $i \geq 2$. Also $x_i \sim x_{i+1}$ for $i \geq 1$. z_1, z_2 have x_1, x_2 as their neighbours. Since $\{x_1, z_1, x_2, z_2, x_1\}$ is a closed path, X is not a tree. For $i \geq 3$, if z is one of the four neighbours of x_i, let $t(x_i, z) = \frac{1}{4}$; and for any $i \geq 2$, $t(y_{i1}, x_i) = t(y_{i2}, x_i) = 1$. For any other pair of neighbours a and b, take $t(a, b) = t(b, a) = 2$. Then, the constants are harmonic on X. Let u be a function defined on X such that $u(x_1) = u(z_1) = u(z_2) = 1$ and if $i \geq 2$, then $u(y_{i1}) = u(y_{i2}) = u(x_i) = i$. Note that $\Delta u(x_1) = \Delta u(z_1) = \Delta u(z_2) = 2$, $\Delta u(x_2) = -4$, and at all other vertices $a \in X$, $\Delta u(a) = 0$. Hence, u is neither subharmonic nor superharmonic on the whole of X. We show now that every positive superharmonic function on X is a constant.

For $i \geq 2$, write $P_i = \{x_i, y_{i1}, y_{i2}\}$. Suppose $s > 0$ is superharmonic on X. Take an arbitrary vertex z in X. Suppose $z \in P_m, m \geq 3$. Let $K = \{x_1, z_1, z_2, x_2, y_{21}, y_{22}\}$ and $\alpha = \inf_{x \in K} s(x)$. Write $E_n = [\bigcup_{n \geq 3} P_n] \backslash K, n \geq 3$. $\partial E_n = \{x_3, x_n\}$. Let $v_n(x) = \alpha \frac{n-u(x)}{n-3}, n > m$. Then $v_n(x)$ is harmonic on E_n and $v_n \leq s$ on ∂E_n, so that by the minimum principle (Corollary 1.4.3), $v_n \leq s$ on E_n. In particular, $s(z) \geq v_n(z) = \alpha \frac{n-u(z)}{n-3}$. Allow $n \to \infty$ to see that $s(z) \geq \alpha$. Since z is an arbitrary vertex in $P_m, m \geq 3$, we conclude that $s(x) \geq \alpha$ for all x in X. However, since s attains its minimum value α on K, by the minimum principle (Proposition 1.4.1), s is the constant α. Since every positive superharmonic function on X is constant, there can be no positive potential on X, that is X is parabolic.

The above example suggests a sufficient condition for an infinite network to be parabolic. Let us say that for a real-valued function f defined on an infinite network X, $\lim_{x \to \infty} f(x) = \infty$, if for any finite number k there exists a finite set A of X such that $f(x) > k$ when $x \in X \backslash A$.

A sufficient condition for an infinite network to be parabolic: Let $u \geq 0$ be a superharmonic function defined outside a finite set A in X, such that $\lim_{x \to \infty} u(x) = \infty$. Then X is parabolic. For, if m is a large positive integer, let E_m denote a finite set such that $u(x) > m$ if $x \notin \overset{0}{E}_m \supset A$. Assume without loss of generality $E_m \subset \overset{0}{E}_n$ if $n > m$. Suppose $s > 0$ is a superharmonic function on X. Let $\alpha = \inf_{x \in E_m} s(x)$. Then, for some $a \in E_m$, $s(a) = \alpha$. Let $z \notin E_m$. Then, for large n, assume $z \in \overset{\circ}{E}_n$ and define $v_n(x) = \alpha \frac{n-u(x)}{n-m}$ for $n > m$. Then, v_n is subharmonic at every vertex on $E_n \backslash \overset{\circ}{E}_m$. Note on ∂E_n, $u(x) > n$ so that $v_n(x) < 0 < s(x)$; and on ∂E_m, $u(x) > m$ so that $v_n(x) < \alpha \leq s(x)$. Hence, by the minimum principle on $E_n \backslash E_m$, we conclude that $s(x) \geq v_n(x)$ on $E_n \backslash E_m$. In particular, $s(z) \geq v_n(z) = \alpha \frac{n-u(z)}{n-m}$. Allow $n \to \infty$ to see that $s(z) \geq \alpha$. Since z is arbitrary in $X \backslash E_m$, we conclude that $s \geq \alpha$ on X. But s attains its minimum on X since $s(a) = \alpha$. Hence s is a constant. Thus, every positive superhamonic function on X is a constant. Hence there cannot be any positive potential on X, that is X is parabolic.

In a parabolic network, it is not possible to define the Green function which is a positive potential. In the classical case in $\Re^n, n \geq 3$, the Newtonian kernel $|x-y|^{n-2}$ is proportional to the Green kernel; but in $\Re^2$, the Green kernel cannot be defined (as a consequence of the Liouville theorem). However, the *logarithmic kernel* $\log \frac{1}{|x-y|}$ in $\Re^2$ has many properties similar to the Green kernel in $\Re^n, n \geq 3$, with the notable exception that the logarithmic kernel is not lower bounded in $\Re^2$. (The terms logarithmic kernel and logarithmic potentials are in vogue since Neumann. If for a Radon measure $\mu \geq 0$, $u(x) = \int_{\Re^2} \log \frac{1}{|x-y|} d\mu(y))$ is superharmonic on $\Re^2$, then u is known as a *logarithmic potential* with associated measure μ.) Associated with the Green kernel in a hyperbolic network is the study of positive potentials which was carried out in the earlier sections. Now, in a parabolic network we shall study pseudo-potentials which correspond to certain superharmonic functions generated by the logarithmic kernel in $\Re^2$. Note that in the study of random walks, a parabolic network is referred to as *recurrent* and the random walk is referred to as *transient* [68] if the Green function can be defined. In fact, a random walk is said to be *recurrent* if the walker is certain to return to the starting point. When a positive probability exists that the walker never returns to the initial position, the random walk is referred to as *transient*. The famous theorem of Polya asserts that a *simple random walk* (a simple random walk on a graph is one in which from any vertex of the graph, there is equal probability of moving to a neighbouring vertex) on a d-dimensional lattice is recurrent for $d = 1, 2$ and transient if $d \geq 3$. *We assume in this section that X is a parabolic network.*

There is little difference in the study of superharmonic functions on a finite subset A of an infinite network, whether the network is hyperbolic or parabolic. For, by using the solution to the Dirichlet problem, we can see that there are positive potentials on any finite set A with non-empty interior even if X is parabolic. So, in this section, we mainly deal with infinite subsets A of a parabolic network X. To start with, here is a version of the *minimum principle special to parabolic networks* even when A is not finite.

Theorem 3.4.1. *Let A be an arbitrary proper subset of a parabolic network X. Let u be a lower bounded superharmonic function on A such that $u \geq \alpha$ on ∂A. Then, $u \geq \alpha$ on A.*

Proof. Let $v = \inf(u, \alpha)$. Then, v is superharmonic on A and $v = \alpha$ on ∂A. Suppose for some $z \in \overset{0}{A}$, $v(z) = \beta < \alpha$. Now, v extended onto X by taking $v = \alpha$ on $X \backslash A$ is superharmonic on X and lower bounded too. Hence v is a constant on X since X is parabolic, $v \equiv \alpha$, a contradiction since $v(z) < \alpha$. □

Remark 3.4.1. 1. The validity of the above Minimum Principle for an infinite subset A is a characteristic property of parabolic networks. For, if X is a hyperbolic network, then a potential $p > 0$ on X is a superharmonic function on X such that $\inf_X p(x) = 0$. Hence, if A is the complement of a finite set, then $\inf_A p(x) = 0$ while $\inf_{\partial A} p(x) > 0$.

2. In Theorem 3.1.7, we have stated the main result on the Dirichlet solution in any arbitrary set F in an infinite network, where the uniqueness of the Dirichlet solution can be asserted if the set F is finite. But if X is parabolic, we can use the above theorem to assert the uniqueness of the Dirichlet solution in any F, when the boundary function is bounded: *Let F be an arbitrary proper subset in a parabolic network and $E \subset \overset{0}{F}$. Let f be a bounded function on $F \backslash E$. Then, there exists a unique bounded function h on F such that $h = f$ on $F \backslash E$ and $\Delta h(x) = 0$ if $x \in E$.*

In Definition 3.2.1, we defined P-sections and S-sections in an infinite network. When X is a parabolic network, if e is a fixed vertex, then every infinite section

$E_i = [e, e_i]$ defined by e is an S-section (Theorem 3.2.4). In particular, there exists an unbounded harmonic function $h_i \geq 0$ on E_i such that $h_i(e) = 0$, $h_i(x) > 0$ and $\Delta h_i(x) = 0$ if $x \neq e$. Choose such a function h_i in each S-section determined by e. Define a function H_e on X such that $H_e = h_i$ in each S-section $E_i = [e, e_i]$ and $H_e = 0$ otherwise. Then, $H_e \geq 0$ is an unbounded function on X with the following properties: $\Delta H_e(x) = c\delta_e(x)$ for x in X and some $c > 0$ (replacing H_e by $c^{-1}H_e$, we can suppose $\Delta H_e(x) = \delta_e(x)$), and $H_e > 0$ outside a finite set A in X. Note that A includes e and all the finite sections determined by e. Clearly, there are many such functions on X. *In the sequel, we shall fix a vertex* e *in* X *and such a function* H_e *in* X. We keep in mind that H_e has many of the important properties of $\log|z|$ in the complex plane. A function similar to H_e has been constructed and used in [5] to define the notion of flux in the context of an axiomatic potential theory without positive potentials.

In a tree T, there are no closed paths and for any pair of vertices u and v, there exists only one (geodesic) path $[u, v]$. Using these facts, for a fixed vertex e in *a recurrent tree* T (that is, there are no positive potentials on T) Bajunaid et al. [15, Sect. 4] have constructed a non-negative function H on T such that $H(e) = 0$ and $\Delta H(x) = \delta_e(x)$ for any non-terminal vertex x in T, with the property that any function h in T which is harmonic outside a finite set is of the form $h = \alpha H + u + b$, where $\Delta u(x) = 0$ at every non-terminal vertex x in T, α is a uniquely determined constant and b is a bounded function on T. This construction (which requires some corrections) relies on the fact that there are no closed paths in T. Note that this function $H(x)$ on T (leaving out the Laplacian on terminal vertices in T) has some of the important (from potential-theoretic view point) properties of $\log|x|$ in $\Re^2$.

Lemma 3.4.2. *Let* h *be a harmonic function defined outside a finite set in* X. *Let* $e \in X$. *Then, there exists a function* u *on* X *such that* $\Delta u(x) = 0$ *if* $x \neq e$ *and* $(u - h)$ *is bounded outside a finite set in* X.

Proof. For a vertex x in X, let $|x|$ denote the distance between e and x (which is the length of the shortest path connecting e and x). Let m be sufficiently large such that h is defined on $\overset{\circ}{B}{}_m^c$ where $B_m = \{x : |x| \leq m\}$. Let f be the bounded Dirichlet solution outside B_m with boundary values h on ∂B_m. Then, $\varphi = h - f$ is harmonic on $\overset{\circ}{B}{}_m^c$ with boundary values 0.

Let $p \geq 0$ be a superharmonic function on B_m with harmonic point singularity e and $p = 0$ on ∂B_m (see Theorem 2.2.12). By the Minimum Principle, $p > 0$ on $\overset{\circ}{B}_m$. For a large $\alpha > 0$, let $v_1 = \varphi$ on $\overset{\circ}{B}{}_m^c$, and $v_1 = \alpha p$ on $\overset{\circ}{B}_m$. On ∂B_m,

$$t(y)v_1(y) = 0 \leq \sum_{y_i \sim y} t(y, y_i)v_1(y_i),$$

since $y \in \partial B_m$ has only one neighbour in $\overset{\circ}{B}_m$ where v_1 can be made to take an arbitrarily large value since α is large and arbitrary. Then, v_1 is subharmonic on

$X\setminus\{e\}$, harmonic outside ∂B_m. Similarly, if $v_2 = \varphi$ on $\overset{\circ}{B}{}_m^c$ and $v_2 = -\alpha p$ on $\overset{\circ}{B}_m$, then v_2 is superharmonic on $X\setminus\{e\}$ and harmonic outside ∂B_m. Choose now $\beta > 0$ large, so that $\alpha p \leq -\alpha p + \beta$ on $\overset{\circ}{B}_m$.

Then on $X\setminus\{e\}$, $v_1 \leq v_2 + \beta$. Hence we can determine a harmonic function u on $X\setminus\{e\}$ such that $v_1 \leq u \leq v_2 + \beta$ on $X\setminus\{e\}$. Clearly, $(u-h)$ is bounded outside $\overset{\circ}{B}_m$. □

Remark 3.4.2. The above Lemma 3.4.2 is not of much use in hyperbolic networks. For, if h is a harmonic function defined outside a finite set, then $h = p_1 - p_2 + H$ outside a finite set, where p_1, p_2 are potentials with finite harmonic support and hence bounded in X and H is harmonic on X (Corollary 3.2.7). Hence, given a harmonic function h outside a finite set in a hyperbolic network, there always exists a harmonic function H on X such that $(H-h)$ is bounded outside a finite set in X.

Definition 3.4.1. A superharmonic function s on an infinite network X is said to be admissible if and only if it has a harmonic minorant outside a finite set.

If s is admissible in a hyperbolic network X, then from what we have just remarked s would have a harmonic minorant on X so that s is a potential up to an additive harmonic function on X. We shall obtain a similar result in the case of a parabolic network, after introducing the notion of pseudo-potentials. The notion of admissible superharmonic functions plays a more crucial role in parabolic networks. So, we continue with our assumption in this section that X is parabolic.

First, remark that if s is an admissible superharmonic function on X and a is an arbitrary vertex in X, then by Lemma 3.4.2, there exists a function u on X such that $\Delta u(x) = 0$ if $x \neq a$, and $s \geq u$ on X. Consequently, we can extend the class of boundary functions which have a Dirichlet solution.

Theorem 3.4.3. *Let s be an admissible superharmonic function on a parabolic network X. Let A be any proper subset of X. Let $h = s$ on ∂A and on the interior of A, let h be the greatest harmonic minorant of s on A. Then h is a solution of the Dirichlet problem on A with boundary values s on ∂A. If in addition s is bounded on ∂A (even if s is not superharmonic on X), then there is a unique bounded solution to the above Dirichlet problem.*

Proof. Let $a \notin A$. Then, there exists u on X such that $\Delta u(x) = 0$ if $x \neq a$, and $(s-u)$ is lower bounded on X. Hence, s has a harmonic minorant on A. Let g be the g.h.m. of s on A. Define $\varphi(x) = s(x)$ on ∂A, and $\varphi(x) = g(x)$ on $\overset{0}{A}$. Then, φ is subharmonic on A, $\varphi \leq s$ on A, and $\varphi = s$ on ∂A. Hence, by Theorem 3.1.7, there exists a harmonic function h on A such that $\varphi \leq h \leq s$ on A, so that $h = s$ on ∂A. Note that h coincides with the g.h.m. g of s on A.

Finally, the assertion about the unique bounded solution when s is bounded on ∂A is a consequence of the more general result proved in Corollary 3.1.9 which states that if f is any bounded function on ∂A, then there exists a unique bounded harmonic function h on A such that $h = f$ on ∂A. □

In Theorem 3.1.10, we discussed the balayage for a non-negative superharmonic function on an infinite network. Since the constants are the only non-negative superharmonic functions on a parabolic network X, this theorem is interesting only in the hyperbolic case. The following theorem is about the balayage of admissible superharmonic functions on a parabolic network. We say that two admissible superharmonic functions are *equivalent superharmonic functions* if the difference between their greatest harmonic minorants outside a finite set is bounded. Note that if p is a positive potential in a hyperbolic network, then the g.h.m. of p outside a finite set A is bounded. For, let h be the g.h.m. of p outside A. Then, the function q equal to p on A and h outside A is a positive superharmonic function on X, majorized by the potential p so that q is a potential on X and it has finite harmonic support A. Since any potential with finite harmonic support is bounded (Domination Principle), q and hence h is bounded. Consequently, if p is a potential and E is an arbitrary set in a hyperbolic network, then p and its balayage R_p^E are potentials which are equivalent in the above sense.

Theorem 3.4.4. *Let s be an admissible superharmonic function on a parabolic network X. Let E be any proper subset of X. Then, there exists an admissible superharmonic function B_s^E on X such that (i) $B_s^E \leq s$ on X, (ii) $B_s^E = s$ on $X \setminus \overset{0}{E}$, (iii) B_s^E is harmonic on E, and (iv) s and B_s^E are equivalent.*

Proof. Fix a vertex $a \notin E$ and let h be the greatest minorant of s such that $\Delta h(x) = 0$ if $x \neq a$. Let u be the (maximal) Dirichlet solution on E with boundary values s on ∂E (Theorem 3.4.3). Let v be the function u on E extended by s on $X \backslash E$. Then, v is superharmonic on X, v is harmonic on E, $v = s$ on $X \setminus \overset{0}{E}$, and $v \leq s$ on X ; since u is the g.h.m. of s on E by construction, $u \geq h$ on E so that $v \geq h$ on $X \setminus \{a\}$. Hence, v is an admissible superharmonic function on X. Let h^* be the g.h.m. of v in $X \setminus \{a\}$, so that $h^* \geq h$. But $v \leq s$, so that $h^* \leq h$. We conclude that v and s have the same g.h.m. in $X \setminus \{a\}$, so v and s are equivalent in X. Hence the theorem is proved if we set $v = B_s^E$. □

Theorem 3.4.5. *For any z in a parabolic network X, there exists a unique admissible superharmonic function $q_z(x)$ with the following properties:*

i. $q_z(x) \leq 0$ *on* X,
ii. $\Delta q_z(x) = -\delta_z(x)$ *for all* x *in* X,
iii. $q_z(z) = 0$, *and*
iv. *for a constant* $\alpha > 0$, $q_z(x) + \alpha H_e(x)$ *is bounded on* X.

The constant α is uniquely determined when the vertex z is fixed. We write $\alpha = \varphi(z)$.

Proof. Take $q_e(x) = -H_e(x)$. As for any other arbitrary vertex z, let A be a finite circled set such that e and z are in $\overset{0}{A}$. Let h be the bounded harmonic function on $X \backslash A$ (Theorem 3.4.3) with values H_e on ∂A, so that $h - H_e \leq 0$ on $X \backslash A$. Let $\Im$ be the family of all superharmonic functions on X such that $s(z) \geq 0$ and $s \geq h - H_e$

on $X \setminus A$. Clearly, the superharmonic function $s \equiv 0$ is in $\Im$. Let $u(x) = \inf_{s \in \Im} s(x)$ for $x \in X$.

For any $y \neq z$, and any $s \in \Im$, if we define $s_1(x) = s(x)$ if $x \neq y$, and $s_1(y) = \sum \frac{t(y,y_i)}{t(y)} s(y_i)$, then $s_1 \leq s$, $s_1(x)$ is harmonic at $x = y$ and $s_1 \in \Im$. Consequently, $\Im$ is a Perron family so that $u(x)$ is harmonic at all $x \neq z$ and $u(x) \geq h(x) - H_e(x)$ on $X \backslash A$. Let $v(x) = h(x) - H_e(x)$ on $X \backslash A$, extended by 0 on A. Then, $v \in \Im$. Hence $u(x) \leq h(x) - H_e(x)$ on $X \backslash A$, and $u(x) \leq 0$ on A. Also, since $y \sim z$ implies that $y \in A$, we have $\sum \frac{t(z,y)}{t(z)} u(y) \leq 0 = u(z)$, so that $u(x)$ is superharmonic at $x = z$. Note that u being upper-bounded and non-constant on X, $u(x)$ cannot be harmonic at $x = z$. Hence $\Delta u(z) < 0$. Write $q_z(x) = -\frac{u(x)}{\Delta u(z)}$. Then, $\Delta q_z(x) = -\delta_z(x)$ for all x in X, $q_z(z) = 0$ and $\left[q_z(x) - \frac{H_e(x)}{\Delta u(z)}\right]$ is a bounded harmonic function on $X \backslash A$ and hence bounded on X. That is, $q_z(x)$ has all the properties stated in the theorem.

To prove the uniqueness of $q_z(x)$, suppose Q is another function on X such that $\Delta Q(x) = -\delta_z(x)$, $Q(z) = 0$ and $(Q + \beta H_e)$ is bounded on X. Then, let $l(x) = q_z(x) - Q(x)$. Since $\Delta l \equiv 0$, then $l(x)$ is harmonic on X. Moreover, $l(x) = (\beta - \alpha) H_e(x) + f(x)$, where $f(x)$ is bounded on X and harmonic outside a finite set. Suppose $\alpha \neq \beta$. Then, since H_e is non-negative, $l(x)$ should be bounded on one side on X. This means that $l(x)$ is a constant, since X is parabolic and $l(x)$ is harmonic and bounded on one side. Consequently, $(\beta - \alpha) H_e$ should be bounded on X. But, this is not possible since H_e is unbounded and $\beta - \alpha \neq 0$. We conclude therefore that $\alpha = \beta$, in which case $l(x)$ is a constant and that constant should be 0, since $l(z) = 0$. □

Proposition 3.4.6. *Let s be a superharmonic function defined outside a finite set in a parabolic network X. Then, there exist two superharmonic functions Q and Q_c on X such that $s = Q - Q_c$ outside a finite set and Q_c has finite harmonic support in X. In the particular case of s being harmonic, Q also has finite harmonic support in X.*

Proof. Choose a finite circled set A such that s is defined on ∂A. Let h be the harmonic function on A with boundary values s on ∂A. Let $p = s$ on $X \backslash A$, and $p = h$ on A. Then $\Delta p = 0$ on $\overset{0}{A}$ and $\Delta p \leq 0$ on $X \backslash A$. It is possible that $\Delta p > 0$ at some vertices on ∂A. Let them be $y_1, y_2, ..., y_i$. Consider $Q(x) = p(x) + \Delta p(y_1) q_{y_1}(x) + ... + \Delta p(y_i) q_{y_i}(x)$, where $q_z(x)$ is the unique superharmonic function with point harmonic support as defined in the above Theorem 3.4.5. Then, $\Delta Q(y_j) = 0$ for $1 \leq j \leq i$, so that Q is superharmonic on X. (Note that $\Delta Q = 0$ on $X \backslash A$, if the given function s is harmonic.) Moreover, if we write $Q_c(x) = \sum_{j=1}^{i} \Delta p(y_j) q_{y_j}(x)$, then Q_c is a superharmonic function with finite harmonic support in X such that $s(x) = p(x) = Q(x) - Q_c(x)$ if $x \in X \backslash A$. □

Proposition 3.4.7. *Let h be a harmonic function defined outside a finite set in a parabolic network X. Then $h = u + \alpha H_e + b$ outside a finite set, where u is a uniquely determined harmonic function on X such that $u(e) = 0$, α is a uniquely determined constant and b is a bounded harmonic function outside a finite set in X.*

Proof. Let Q be a superharmonic function on X having harmonic support only at a finite set of vertices $z_1, z_2, ..., z_m$. Define $u^\bullet(x) = Q(x) + \sum_{j=1}^{m} [\Delta Q(z_j)] q_{z_j}(x)$ on X. Then $\Delta u^\bullet(x) = 0$ for all x in X, that is $u^\bullet$ is harmonic on X. Now, by construction, for each j there exists a constant α_j such that $(q_{z_j} + \alpha_j H_e)$ is bounded harmonic outside a finite set. Write $\alpha^\bullet = \sum_{j=1}^{m} [\Delta Q(z_j)]\alpha_j$. Then, $u^\bullet(x) + \alpha^\bullet H_e(x) = Q(x)+$ (a bounded harmonic function) outside a finite set. Hence, $Q(x) = u^\bullet(x)+\alpha^\bullet H_e(x)+b^\bullet(x)$ outside a finite set where $b^\bullet(x)$ is bounded harmonic outside a finite set. Using Proposition 3.4.6 and a similar representation for Q_c, we can now write $h(x) = u_1(x)+\alpha H_e(x)+b_1(x)$ outside a finite set, where u_1 is harmonic on X and b_1 is bounded harmonic. Write $u(x) = u_1(x) - u_1(e)$ and $b = b_1 + u_1(e)$. Then, $h(x) = u(x) + \alpha H_e + b$ outside a finite set.

To show that u and α are uniquely determined, suppose

$$h(x) = v(x) + \beta H_e(x) + b_2(x)$$

is another such representation outside a finite set. Then,

$$(\alpha - \beta)H_e(x) = [v(x) - u(x)] + [b_2(x) - b(x)]$$

outside a finite set. If $\alpha - \beta \neq 0$, then we arrive at a contradiction as shown in the proof of Theorem 3.4.5. Then, $v - u = b - b_2$ outside a finite set and hence bounded on X. Since $v - u$ is bounded and harmonic on X, it must be a constant c. Then, $c = v(e) - u(e) = 0$, so that $v = u$ on X and consequently $b = b_2$ also. □

Let $E_k = \{x : |x| = d(x, e) \le k\}$. Let $D_k f$ denote the Dirichlet solution on E_k with boundary values f on ∂E_k. Suppose f is a function defined outside a finite set. Then for any $x \in X$, $\{D_k f(x)\}$ is well defined for large k. If $f(x)$ is a bounded function on X, then $\{D_k f(x)\}$ is bounded for any x; if $h(x)$ is a harmonic function on X, then $h(x) = D_k h(x)$ when $x \in \overset{\circ}{E}_k$; if $s(x)$ is a non-harmonic superharmonic function on X, then for any x, $\lim_{k\to\infty} D_k s(x) = -\infty$; and for any x, $\lim_{k\to\infty} D_k H_e(x) = \infty$.

Corollary 3.4.8. *Let h be a harmonic function defined outside a finite set in X. Then, there exists a harmonic function u on X such that $(h - u)$ is bounded outside a finite set if and only if the sequence $\{D_k h(x)\}$ for large k is bounded at some, and hence every vertex x in X.*

Proof. By Proposition 3.4.7, $h = u + \alpha H_e + b$ outside a finite set. Since $D_k u(x) = u(x)$ for all large k, $\lim_{k\to\infty} D_k H_e \equiv \infty$, and the sequence $\{D_k b(x)\}$ for large k, is bounded for any $x \in X$, we find that $\{D_k h(x)\}$ is bounded for some x if and only if $\alpha = 0$; that is, if and only if $(h - u)$ is bounded outside a finite set. □

3.5 Flux at Infinity

Let Ω be an open set in the Euclidean space $\Re^3$. Let ω be a bounded domain with smooth boundary $\partial\omega$ such that $\varpi \subset \Omega$. Let $\vec{v}$ be a continuously differentiable function on Ω. Then the Green's formula can be presented [13, p.4] in the form

$$\int_{\omega} div\, \vec{v}\, d\sigma = \int_{\partial\omega} \vec{v}\,.\,\vec{n}\, ds,$$

where $\vec{n}$ is the outer normal. Suppose f and g are C^2-functions on Ω.

Take $\vec{v} = f\nabla g$. Then

$$\int_{\omega} [f\Delta g + (grad\ f, grad\ g)]d\sigma = \int_{\partial\omega} f\frac{\partial g}{\partial n^+} ds,$$

where $\frac{\partial g}{\partial n^+}$ denotes the outer normal derivative of g. In particular, $\int_{\omega} \Delta g d\sigma = \int_{\partial\omega} \frac{\partial g}{\partial n^+} ds$ and the right side integral is known as the *outer flux* corresponding to g and ω. The notion of flux is important in the study of many physical problems (gravitation, electricity). Suppose μ is a mass distribution on a compact set $K \subset \omega$. Then the (Newton) potential in $\mathbb{R}^3$ due to μ is $g(x) = \int_K |x-y|^{-1} d\mu(y)$ and $\Delta g(x) = -4\pi d\mu(x)$ *in the sense of distributions.* Consequently, if g is a C^2-function, then the *inner flux* $\int_{\partial\omega} \frac{\partial g}{\partial n^-} ds = 4\pi$ (total mass).

Suppose now that h is a harmonic function defined outside a compact set in $\Re^n, n \geq 2$. Then, $\int_{|s|=R} \frac{\partial h}{\partial n^+} ds$ is independent of R for large values of R. This constant is known as the *flux at infinity* of the harmonic function h, denoted by $flux_\infty h$. If b is bounded harmonic outside a finite set, then $\lim_{|x|\to\infty} b(x)$ exists. In $\Re^2$, $flux_\infty b = 0$; however in $\Re^n, n \geq 3$, $flux_\infty b$ may not be 0 as in the case of $b(x) = |x|^{2-n}$. This indicates a possibility of using the notion of flux at infinity to distinguish between hyperbolic and parabolic networks.

Let u be a superharmonic function defined on $|x| > R$. Let $R < r_1 < r$. Let $h(x)$ be the Dirichlet solution on $r_1 < |x| < r$ with boundary values u. Then $\lim_{r_1\to r} \int_{|s|=r_1} \frac{\partial h}{\partial n^+}(s) ds$ exists and is known as (up to a multiplicative constant depending on the dimension of the space) the *inner flux* f_i of u on $|x| = r$. A similar procedure with respect to $r < r_1^{'}$ defines the *outer flux* f_e of u on $|x| = r$. Now both f_i and f_e have a common limit α when $r \to \infty$. This common value α is known as the *flux at infinity of the superharmonic function* u defined outside a compact set.

In a related development, Brelot [27, p.194] shows that if h is a harmonic function defined outside a compact set in the Euclidean plane, then it has a representation

$$h(x) = C + \alpha \log \rho + \sum_{n=1}^{\infty} a_n(\theta)\rho^n + \sum_{n=1}^{\infty} a_n^{'}(\theta)\rho^{-n}$$

outside a compact set where $x = (\rho, \theta)$ in polar coordinates, C is a constant, α is the $flux_\infty h$, and $a_n(\theta), a_n^{'}(\theta)$ are of the form $\lambda_n \cos n\theta + \mu_n \sin n\theta$. To obtain an analogue of this very useful representation of a harmonic function defined outside a compact set in an axiomatic potential theory without positive potentials on a locally compact space X, one starts with the construction [5, pp.112–114] of an unbounded non-negative harmonic function H outside a compact set in X. This function H has many of the basic properties of $\log |x|$ outside the unit disc in the plane. Then, it is proved that any harmonic function h outside a compact set in X is of the form $h(x) = \alpha H(x) + u(x) + b(x)$ where α is a uniquely determined constant, u is harmonic on X and b is bounded harmonic outside a compact set in X. This uniquely determined constant α is defined to be the *flux at infinity of h,* thus avoiding the explicit use of normal derivatives in the definition of flux. The same procedure as in the locally compact case can be adopted to define the flux at infinity of a harmonic function defined outside a finite set in a parabolic network [9, pp.13–15]. A similar method has been used in Cohen et al. [37] to study the flux at infinity of a superharmonic function defined outside a finite set in a Cartier tree.

Recall that in Proposition 3.4.7, for a harmonic function h defined outside a finite set in a parabolic network X, we have obtained a representation of the form $h = u + \alpha H_e + b$, where u is harmonic on X, b is bounded harmonic outside a finite set and α is a uniquely determined constant. Let h be a harmonic function defined outside a finite set on a parabolic network X. Define the $flux$ *of h at infinity* $= flux_\infty h = \alpha$. Then:

1. If h is any harmonic function defined outside a finite set, then $flux_\infty h = 0$ if and only if there exists a harmonic function u on X such that $|u - h|$ is bounded outside a finite set.
2. If b is a bounded harmonic function defined outside a finite set, then $flux_\infty b = 0$.
3. If h_1 and h_2 are two harmonic functions defined outside a finite set, then $flux_\infty(\alpha_1 h_1 + \alpha_2 h_2) = \alpha_1 flux_\infty h_1 + \alpha_2 flux_\infty h_2$, for any real numbers α_1 and α_2.

Recall (Theorem 3.4.5) that for any z in X, there exists a unique admissible superharmonic function $q_z(x) \leq 0$ on X and a unique positive number $\varphi(z)$ such that $\Delta q_z(x) = -\delta_z(x)$ for all x in X, $q_z(z) = 0$ and $[q_z + \varphi(z)H_e]$ is bounded on X. Note that $flux_\infty H_e = 1$, $flux_\infty q_z = -\varphi(z)$, and $\varphi(e) = 1$.

Let A be a finite set such that $\Delta h = 0$ at every vertex outside A. Extend h as a function v on X. Let $u(x) = v(x) + \sum_{y \in A} \Delta v(y) q_y(x)$. Then, $\Delta u(x) = 0$ for every x in X, that is u is harmonic on X. Moreover, outside a finite set,

$$v(x) = u(x) - \sum_{y \in A} \Delta v(y)[q_y(x) + \varphi(y) H_e(x)] + \left[\sum_{y \in A} \varphi(y) \Delta v(y) \right] H_e(x),$$

which can be written as

$$v(x) = u(x) + b(x) + \alpha H_e(x),$$

where $[\sum_{y \in A} \varphi(y) \Delta v(y)] = \alpha$, and $b(x) = -\sum_{y \in A} \Delta v(y)[q_y(x) + \varphi(y) H_e(x)]$ is bounded harmonic outside a finite set, since $[q_y(x) + \varphi(y) H_e(x)]$ is bounded on X and A contains only a finite number of vertices. Hence,

$$flux_\infty h = \left[\sum_{y \in A} \varphi(y) \Delta v(y) \right].$$

This provides an alternate way of defining $flux_\infty h$, without recourse to the decomposition of h given in Proposition 3.4.7. If f is a real-valued function on X, equal to a harmonic function h outside a finite set in X, then take $flux_\infty f = flux_\infty h$. In particular, if s is a superharmonic function with finite harmonic support, then we can write $flux_\infty s = [\sum_{y \in X} \varphi(y) \Delta s(y)]$.

In the Euclidean space $\Re^2$, it can be proved that if s is a superharmonic function on $|x| > R$, then $flux_\infty s$ is finite if and only if s has a harmonic minorant and if h is the greatest harmonic minorant of s on $|x| > R$, then $flux_\infty s = flux_\infty h$. As a discrete analogue of this result, we propose the following definition.

Definition 3.5.1. Let v be a superharmonic function defined on an infinite network X. If v is not admissible, define the flux at infinity of v as $-\infty$; if v is admissible and if h is the g.h.m. of u in $X \setminus \{e\}$, then define $flux_\infty v = flux_\infty h$. If u is a subharmonic function on X, define $flux_\infty u = -flux_\infty(-u)$.

Consequently,

1. If s and v are two superharmonic functions on X such that s and v are equivalent, (that is the difference of the g.h.m. of s and v outside a finite set on X is bounded, see Theorem 3.4.4), then $flux_\infty s = flux_\infty v$.
2. If v_1 and v_2 are two superharmonic functions on X and if α_1, α_2 are two non-negative constants, then $flux_\infty(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 flux_\infty v_1 + \alpha_2 flux_\infty v_2$. For, let h_1 and h_2 be the g.h.m. of v_1 and v_2 respectively in $X \setminus \{e\}$. Let h be the g.h.m. of $\alpha_1 v_1 + \alpha_2 v_2$ in $X \setminus \{e\}$. Since the subharmonic function $h - \alpha_2 v_2 \leq$

$\alpha_1 v_1$ on $X \setminus \{e\}$, and since $\alpha_1 h_1$ is the g.h.m. of $\alpha_1 v_1$ on $X \setminus \{e\}$, we find that $h - \alpha_2 v_2 \leq \alpha_1 h_1$ on $X \setminus \{e\}$. Since $h - \alpha_1 h_1 \leq \alpha_2 v_2$ on $X \setminus \{e\}$, we find as before $h - \alpha_1 h_1 \leq \alpha_2 h_2$ on $X \setminus \{e\}$ so that $h \leq \alpha_1 h_1 + \alpha_2 h_2$ on $X \setminus \{e\}$. But h is the g.h.m. of $\alpha_1 v_1 + \alpha_2 v_2$ on $X \setminus \{e\}$; this implies that $h = \alpha_1 h_1 + \alpha_2 h_2$. Further, $flux_\infty h = \alpha_1 flux_\infty h_1 + \alpha_2 flux_\infty h_2$.

3. If v is a non-harmonic superharmonic function defined on X, then $flux_\infty v < 0$. For, let h be any harmonic function outside a finite set such that $h \leq v$. Write $h = u + \alpha H_e + b$ as before. Suppose $\alpha \geq 0$. Then, the superharmonic function $(v - u)$ is lower bounded outside a finite set. X being parabolic, this implies that $(v - u)$ is a constant, that is v is harmonic on X, a contradiction.
4. If v is a superharmonic function on X, then $flux_\infty v = 0$ if and only if v is harmonic.

Theorem 3.5.1. *Let v be a superharmonic function on a parabolic network X. Let $\varphi(x)$ be as defined in Theorem 3.4.5. Then $flux_\infty v \leq \sum_{x \in X} \varphi(x) \Delta v(x)$.*

Proof. If v is not admissible, then by definition $flux_\infty v = -\infty$ and the theorem is trivial. Let us assume therefore that v is admissible, that is $flux_\infty v$ is finite. Let$\{E_m\}$ be an exhaustion of X by an increasing sequence of finite connected circled sets. Define

$$v_m(x) = v(x) + \sum_{y \in E_m} \Delta v(y) q_y(x).$$

Then, for $y \in E_m$, $\Delta v_m(y) = \Delta v(y) - \Delta v(y) = 0$ and for $y \notin E_m$, $\Delta v_m(y) = \Delta v(y) \leq 0$. Hence v_m is superharmonic on X and is admissible, so that $flux_\infty v_m \leq 0$. Hence,

$$flux_\infty v + \sum_{y \in E_m} \Delta v(y) flux_\infty q_y(x) \leq 0.$$

Now, $q_y(x) + \varphi(y) H_e(x)$ is bounded on X and hence $flux_\infty q_y(x) = -\varphi(y)$, since by definition $flux_\infty H_e = 1$. Consequently, $flux_\infty v - \sum_{y \in E_m} \varphi(y) \Delta v(y) \leq 0$.

Since $\Delta v \leq 0$, $\sum_{y \in E_m} \varphi(y) \Delta v(y)$ is a decreasing sequence in m. Allow $m \to \infty$, and conclude $flux_\infty v \leq \sum_{y \in X} \varphi(y) \Delta v(y)$. □

Theorem 3.5.2. *Let u be a superharmonic function on a parabolic network X with symmetric conductance. If $\sum_{x \in X} \Delta u(x)$ is finite, then u is admissible and $flux_\infty u \geq \sum_{x \in X} \Delta u(x)$.*

Proof. Given that for the superharmonic function u, $\sum_{x \in X} \Delta u(x)$ is finite. Let $\{E_m\}$ be an exhaustion of X by an increasing sequence of finite connected circled sets such that $e \in \overset{\circ}{E}_1$. Let h_m be the Dirichlet solution in E_m with boundary values u on $\partial E_m \cup \{e\}$. Clearly, $\frac{\partial u}{\partial n^-} \geq \frac{\partial h_m}{\partial n^-}$ on ∂E_m. Hence,

$$\begin{aligned} -\infty < \sum_{x \in X} \Delta u(x) &= -\lim_{m \to \infty} \sum_{s \in \partial E_m} \frac{\partial u}{\partial n^-}(s) \\ &\leq -\lim_{m \to \infty} \sum_{s \in \partial E_m} \frac{\partial h_m}{\partial n^-}(s) \\ &= -\lim_{m \to \infty} [-\Delta h_m(e)] \\ &= \lim_{m \to \infty} [\Delta h_m(e)]. \end{aligned}$$

Remark that h_m is a decreasing sequence and write $v = \lim_{m \to \infty} h_m$. Then, h_m being superharmonic on E_m, either $v \equiv -\infty$ or v is superharmonic on X such that $\Delta v(x) = 0$ if $x \neq e$. Suppose $v \equiv -\infty$. Then, given $\lambda > 0$, there exists m such that $h_m(x_0) < -\lambda$ for some $x_0 \sim e$. However, note that $h_m(e) = u(e)$ for all m. Hence,

$$\begin{aligned} \Delta h_m(e) &= \sum_{x \in X} t(e,x)[h_m(x) - h_m(e)] \\ &= \sum_{x \neq x_0} t(e,x)[h_m(x) - h_m(e)] + t(e,x_0)[h_m(x_0) - h_m(e)] \\ &< \sum_{x \neq x_0} t(e,x)[h_1(x) - u(e)] + t(e,x_0)[-\lambda - u(e)] \\ &= -\lambda t(e,x_0) + (\text{a constant independent of } m) \\ &\leq -\lambda\beta + (\text{ a constant independent of } m), \text{ where } \beta = \inf_{x \sim e} t(x,e) > 0. \end{aligned}$$

This means that $\lim_{m \to \infty} \Delta h_m(e) = -\infty$, contradicting what we have proved earlier. Hence, v is superharmonic on X. Moreover, if $x \neq e$, then $v(x) \leq u(x)$ and $\Delta v(x) = 0$. Note that if h is harmonic on $X \backslash \{e\}$ and $h(x) \leq u(x)$ if $x \neq e$, then $h \leq h_m$ on E_m, so that $h \leq v$ on $X \backslash \{e\}$. Hence, v is the g.h.m. of u in $X \backslash \{e\}$, so that

$$flux_\infty u = flux_\infty v = \varphi(e)\Delta v(e) = \Delta v(e).$$

But $\Delta v(e) = \lim_{m \to \infty} \Delta h_m(e) \geq \sum_{x \in X} \Delta u(x)$. Hence, $flux_\infty u \geq \sum_{x \in X} \Delta u(x)$. □

Corollary 3.5.3. *Let u be a superharmonic function on a parabolic network X such that $\sum_{x\in X} \Delta u(x)$ is finite. Assume that the conductance in X is symmetric. Then u is admissible and $\sum_{x\in X} \Delta u(x) \leq flux_\infty u \leq \sum_{x\in X} \varphi(x)\Delta u(x)$, where $\varphi(x)$ is as defined in Theorem 3.4.5.*

Proof. This is a consequence of Theorems 3.5.1 and 3.5.2. □

An improvement of the above corollary is possible if we assume that there are no cycles (= closed paths) in the parabolic network X with symmetric conductance. Actually in this case we show that $\varphi(x) = 1$ for each x in X. Recall that for a fixed vertex e, $H_e \geq 0$ is an associated unbounded function on X such that $\Delta H_e(x) = \delta_e(x)$ for each x in X, $H_e > 0$ outside a finite set and $H_e(e) = 0$.

Lemma 3.5.4. *Let X be a parabolic network without any cycles. Assume that X has symmetric conductance. Then, for any z in X, there exists a unique function $q_z(x)$ on X such that $\Delta q_z(x) = -\delta_z(x)$ for all x in X, $q_z(z) = 0$ and $(q_z + H_e)$ is bounded on X.*

Proof. Let $q_e(x) = -H_e(x)$. Let $a \sim e$. Let A be the set of all vertices x such that the (geodesic) path joining x to e passes through a. In particular, $a \in A$ and $e \notin A$. We show now that a constant α can be chosen so that the function s defined on X such that $s(x) = q_e(x)$ if $x \notin A$ and $s(x) = q_e(x) + \alpha$ if $x \in A$ satisfies the relation $\Delta s(x) = -\delta_a(x)$.

First note that except e, no vertex $y \in X\backslash A$ can have a neighbour in A. For, if $y \sim b \in A$, then there exists a cycle $\{e, ..., y, b, ..., a, e\}$ which is not possible because of the assumption that there are no closed paths in X. Consequently, for any x in X different from e and a, $\Delta s(x) = 0$. Now

$$\begin{aligned}\Delta s(e) &= \sum_{x\sim e} t(e,x)[s(x) - s(e)]\\ &= \sum_{x\sim e} t(e,x)s(x), \text{since } s(e) = q_e(e) = 0\\ &= \sum_{e\sim x\in X\backslash A} t(e,x)q_e(x) + t(e,a)s(a).\end{aligned}$$

Note $\sum_{x\sim e} t(e,x)q_e(x) = \Delta q_e(e) = -1$, that is $\sum_{e\sim x\in X\backslash A} t(e,x)q_e(x) + t(e,a)q_e(a) = -1$. Consequently,

$$\begin{aligned}\Delta s(e) &= [-1 - t(e,a)q_e(a)] + t(e,a)s(a)\\ &= [-1 - t(e,a)q_e(a)] + t(e,a)[q_e(a) + \alpha]\\ &= -1 + \alpha t(e,a).\end{aligned}$$

Hence, to satisfy the condition $\Delta s(e) = 0$, we should choose $\alpha = [t(e,a)]^{-1}$. With this value of α,

$$\begin{aligned}\Delta s(a) &= \sum_{x\sim a} t(a,x)[s(x) - s(a)] \\ &= \sum_{a\sim x\in A} t(a,x)[q_e(x) - q_e(a)] + t(a,e)[s(e) - s(a)] \\ &= \sum_{a\sim x\in A} t(a,x)[q_e(x) - q_e(a)] + t(a,e)[q_e(e) - (q_e(a) + \alpha)] \\ &= \sum_{x\sim a} t(a,x)[q_e(x) - q_e(a)] - t(a,e)\alpha \\ &= \Delta q_e(a) - 1 \text{ since } t(a,e)\alpha = t(e,a)\alpha = 1 \\ &= 0 - 1 = -1.\end{aligned}$$

Thus, $\Delta s(x) = -\delta_a(x)$ for all x in X; since $|s - q_e| \leq \alpha$ and $q_e(x) = -H_e(x)$, we conclude that $s + H_e$ is bounded on X. Define $q_a(x) = s(x) - s(a)$. Then, $q_a(a) = 0$, $\Delta q_a(x) = -\delta_a(x)$ for all x and $(q_a + H_e)$ is bounded on X.

Thus we have proved that for any $a \sim e$, there exists a function $q_a(x)$ on X such that $\Delta q_a(x) = -\delta_a(x)$ for all x, $q_a(a) = 0$, and $(q_a - q_e)$ is bounded on X. This procedure can be repeated for any neighbour of a. Since X is connected, any vertex z in X can be joined to e by a finite arc, so that we can determine a function $q_z(x)$ on X such that $\Delta q_z(x) = -\delta_z(x)$ for $x \in X$, $q_z(z) = 0$, and $q_z - q_e = q_z + H_e$ is bounded on X.

To prove the uniqueness, suppose u is a function on X such that $\Delta u(x) = -\delta_z(x)$, $u(z) = 0$, and $u + H_e$ is bounded on X. Then, take $v(x) = u(x) - q_z(x)$. Since $\Delta v(x) = 0$ for all x in X, v is harmonic on X; moreover, $v = (u + H_e) - (q_z + H_e)$ is bounded on X. Hence v is a constant; since $v(z) = 0$, we conclude that $v \equiv 0$. □

The above lemma says that if X is a parabolic network without cycles but has symmetric conductance, then we can choose $q_z(x)$ in Theorem 3.4.5 in a manner that $\varphi(x) = 1$ for every x in X.

Theorem 3.5.5. *Let v be a superharmonic function on a parabolic network X without cycles. Assume X has symmetric conductance. Then v is admissible if and only if $\sum_{x\in X} \Delta v(x)$ is finite. In fact*

$$flux_\infty v = \sum_{x\in X} \Delta v(x) = -\lim_{m\to\infty} \sum_{s\in\partial E_m} \frac{\partial v}{\partial n^-}(s),$$

where $\{E_m\}$ is an exhaustion of X by an increasing sequence of finite circled sets, (that is, each E_m is finite and circled, $E_m \subset \overset{\circ}{E}_{m+1}$ and $X = \bigcup_m E_m$).

Proof. From Theorem 3.5.2, if $\sum_{x\in X} \Delta v(x)$ is finite, then v is admissible. Conversely, if v is admissible, then $flux_{\infty}v$ is finite and from Theorem 3.5.1 (with $\varphi \equiv 1$), $\sum_{x\in X} \Delta v(x)$ is finite. Hence, if v is not admissible, then $flux_{\infty}v = \sum_{x\in X} \Delta v(x) = -\infty$. Let us assume therefore that v is admissible. Then again Corollary 3.5.3 shows that

$$\sum_{x\in X} \Delta v(x) \leq flux_{\infty}v \leq \sum_{x\in X} \Delta v(x).$$

The second part follows from the Green's formula $\sum_{x\in \overset{0}{E}_m} \Delta v(x) = -\sum_{s\in\partial E_m} \frac{\partial v}{\partial n^-}(s)$.

□

Remark 3.5.1. From the above theorem, it can be seen that if v is a superharmonic function on a parabolic network X without cycles but with symmetric conductance, then

$$flux_{\infty}v = \lim_{m\to\infty} \sum_{x\in E_m} \Delta v(x) = -\lim_{m\to\infty} \sum_{s\in\partial E_m} \frac{\partial v}{\partial n^-}(s).$$

We can use this expression for $flux_{\infty}v$ to extend the definition of flux at infinity to superharmonic functions defined outside a finite set in a parabolic network without cycles, but having symmetric conductance. Let v be a superharmonic function defined outside a finite set in a parabolic network X without cycles. Then as in Proposition 3.4.6, there exist two superharmonic functions Q and Q_c on X such that $v = Q - Q_c$ outside a finite set and Q_c has finite harmonic support so that $flux_{\infty}Q_c$ is finite. Define

$$flux_{\infty}v = flux_{\infty}Q - flux_{\infty}Q_c.$$

If $v = Q^{'} - Q_c^{'}$ is another such representation, then $Q + Q_c^{'} = Q^{'} + Q_c$ outside a finite set, so that $flux_{\infty}(Q + Q_c^{'}) = flux_{\infty}(Q^{'} + Q_c)$. Hence, $flux_{\infty}Q + flux_{\infty}Q_c^{'} = flux_{\infty}Q^{'} + flux_{\infty}Q_c$. This implies, since $flux_{\infty}Q_c$, $flux_{\infty}Q_c^{'}$ are finite, $flux_{\infty}Q - flux_{\infty}Q_c = flux_{\infty}Q^{'} - fluxQ_c^{'}$. Hence, there is no ambiguity in the definition of $flux_{\infty}v$. Further,

$$flux_{\infty}v = -\lim_{m\to\infty} \sum_{s\in\partial E_m} \frac{\partial v}{\partial n^-}(s).$$

This definition accords with the earlier definition for a harmonic function defined outside a finite set, as seen in the following theorem.

Theorem 3.5.6. *Let X be a parabolic network without any cycles but with symmetric conductance. If $h = u + \alpha H_e + b$ is a harmonic function defined outside a finite set A in X, then $flux_{\infty}h = \alpha = -\sum_{s\in\partial F} \frac{\partial h}{\partial n^-}(s)$, where F is any large finite set such that $A \subset \overset{0}{F}$.*

Proof. Let A be a finite set such that $\Delta h = 0$ at every vertex outside $\overset{0}{A}$. Extend h as a function v on X by taking the Dirichlet solution in A with boundary values h. Let $u(x) = v(x) + \sum_{y\in\partial A} \Delta v(y) q_y(x)$. Then, $\Delta u(x) = 0$ for every x in X, that is u is harmonic on X. Moreover, outside a finite set,

$$v(x) = u(x) - \sum_{y\in\partial A} \Delta v(y)\,[q_y(x) + H_e(x)] + \left[\sum_{y\in\partial A} \Delta v(y)\right] H_e(x),$$

where $\sum_{y\in\partial A} \Delta v(y) = \alpha$, and $b(x) = -\sum_{y\in\partial A} \Delta v(y)[q_y(x) + H_e(x)]$ is bounded harmonic outside a finite set, since $[q_y(x) + H_e(x)]$ is bounded on X for each y and ∂A contains only a finite number of vertices. Since $v = h$ and $\Delta v = 0$ outside a finite set in X, if $A \subset \overset{0}{F}$ then

$$\alpha = \sum_{y\in\partial A} \Delta v(y) = \sum_{y\in \overset{0}{F}} \Delta v(y) = -\sum_{s\in\partial F} \frac{\partial v}{\partial n^-}(s) = -\sum_{s\in\partial F} \frac{\partial h}{\partial n^-}(s).$$

□

Corollary 3.5.7. *Let X be an infinite network without any cycles but with symmetric conductance. Then X is parabolic if and only if for any bounded harmonic function h defined outside a finite set $\sum_{s\in\partial F} \frac{\partial h}{\partial n^-}(s) = 0$ for any large finite set F.*

Proof. For, if X is hyperbolic and if p is a positive potential in X with finite harmonic support A, then by the Domination Principle $p(x) \leq \max_{z\in A} p(z)$ for any $x \in X$, and for any finite set F, $A \subset \overset{0}{F}$, $\sum_{s\in\partial F} \frac{\partial p}{\partial n^-}(s) = -\sum_{x\in \overset{0}{F}} \Delta p(x) = -\sum_{x\in A} \Delta p(x) \neq 0$. On the other hand, in a parabolic network, for any bounded harmonic function h defined outside a finite set, $flux_\infty h = -\sum_{s\in\partial F} \frac{\partial h}{\partial n^-}(s) = 0$.

□

Remark 3.5.2. In a parabolic network X, for any z in X, there exists by Theorem 3.4.5, a unique function $q_z(x)$ and a unique constant $\varphi(z)$ such that $q_z(z) = 0$, $\Delta q_z(x) = -\delta_z(x)$ and $[q_z(x) + \varphi(z) H_e(x)]$ is bounded on X. We shall now give a characterization to determine the constant $\varphi(z)$ when X is a tree T. Recall that in a tree T, the symmetry of conductance is not guaranteed.

Consequently, in the proof of Lemma 3.5.4, in the construction of the superharmonic function s, we obtain $\alpha = [t(e,a)]^{-1}$ and $\Delta s(a) = 0 - \frac{t(a,e)}{t(e,a)}$ if we are dealing in a tree T without symmetric conductance. Hence $\Delta s(x) = -\frac{t(a,e)}{t(e,a)} \delta_a(x)$ for all $x \in T$. Then, writing

$$q_a(x) = \frac{t(e,a)}{t(a,e)}\left[s(x) - s(a)\right]$$

we find $q_a(a) = 0$ and $\Delta q_a(x) = -\delta_a(x)$ for all x in T and $\left[q_a(x) + \frac{t(e,a)}{t(a,e)} H_e(x)\right]$ is bounded on T.

Let us repeat the procedure with respect to a vertex $b \sim a$, to construct $q_b(x)$ such that $q_b(b) = 0$ and $\Delta q_b(x) = -\delta_b(x)$ for all x in T and

$$\left[q_b(x) + \frac{t(e,a)}{t(a,e)} \cdot \frac{t(a,b)}{t(b,a)} H_e(x)\right]$$

is bounded on T. Let now z be an arbitrary vertex in T. Then, there is a unique path $\{e, z_1, z_2, ..., z_n = z\}$ joining e to z. Repeating the above procedure, we construct $q_z(x)$ on T such that and $\Delta q_z(x) = -\delta_z(x)$ for all x in T and

$$\left[q_z(x) + \frac{t(e,z_1)}{t(z_1,e)} \cdot \frac{t(z_1,z_2)}{t(z_2,z_1)} ... \frac{t(z_{n-1},z)}{t(z,z_{n-1})} H_e(x)\right]$$

is bounded on T. From the uniqueness of such a function $q_z(x)$ we conclude

$$\varphi(z) = \frac{t(e,z_1)t(z_1,z_2)...t(z_{n-1},z)}{t(z,z_{n-1})t(z_{n-1},z_{n-2})...t(z_1,e)}.$$

3.6 Pseudo-Potentials

In the classical potential theory in $\Re^2$, if $\mu \geq 0$ is a Radon measure with compact support K, then $U^\mu(x) = \int_{\Re^2} \log \frac{1}{|x-y|} d\mu(y)$ is a superharmonic function on $\Re^2$, harmonic outside K. For some Radon measure $\mu \geq 0$, $U^\mu(x) = \int_{\Re^2} \log \frac{1}{|x-y|} d\mu(y)$ is still a superharmonic function in $\Re^2$, even if μ does not have compact support. When $U^\mu(x)$ is superharmonic on $\Re^2$, we call it the *logarithmic potential* with associated measure μ. This happens if and only if $\int_{|x|>1} \log|x|\, d\mu(x)$ is finite. In particular, if $U^\mu(x)$ is a logarithmic potential, then $\int_{\Re^2} d\mu(x)$ is finite, that is $U^\mu(x)$ is an admissible superharmonic function. But not every admissible superharmonic function is a logarithmic potential. Logarithmic potentials are very useful when we study in $\Re^2$, single layer and double layer potentials, analyticity of harmonic functions, Poisson integrals, Harnack inequalities etc. [27, pp.176–186]. Since logarithmic potentials form only a small subclass of admissible superharmonic functions, it is convenient to introduce the notion of *pseudo-potentials* in the larger class of admissible superharmonic functions and prove that any admissible superharmonic function in $\Re^2$ is the sum of a pseudo-potential and a harmonic function.

Actually, such decompositions for admissible superharmonic functions in the Brelot axiomatic case without positive potentials have been given in [7] by using H-functions: Let Ω be a Brelot harmonic space without positive potentials. Let $H \geq 0$ be a fixed harmonic function defined outside a compact set K in Ω, tending to 0 on ∂K with $flux_\infty H = 1$. Let s be an admissible superharmonic function on Ω with $flux s = \alpha$ and let h be the greatest harmonic minorant of s outside K; then s is said to be an H-function [7] if and only if $(h - \alpha H)$ is bounded in a neighbourhood of the Alexandroff point at infinity of Ω. (These H-functions have many similarities with *subharmonic functions of potential-type*; this latter class of functions was introduced in the Euclidean space $\Re^2$ by Arsove [12], by making use of the characteristic function and the order associated with any subharmonic function in $\Re^2$.) Then it is proved [7, Proposition 1] that any admissible superharmonic function can be written as the sum of an H-function and a harmonic function on Ω. This decomposition is unique up to an additive constant. Bajunaid et al. [16] call an admissible superharmonic function s on Ω an *H-potential* if $\liminf_{x\to\infty}[h(x) - \alpha H(x)] = 0$ where the *lim inf* is taken with respect to the Alexandroff one-point compactification of Ω. Then they prove [16, Theorem 3.1] that every admissible superharmonic function on Ω can be written uniquely as the sum of an H-potential and a harmonic function. Earlier Cohen et al. [37] have considered the discrete analogue of this decomposition in the context of trees without positive potentials.

In this section, we define pseudo-potentials in a parabolic network X and obtain some of their properties including their role in the Riesz-type decomposition of admissible superharmonic functions in X.

Definition 3.6.1. An admissible superharmonic function q on a parabolic network X is said to be a pseudo-potential if and only if its g.h.m. on $X \setminus e$ is of the form $\alpha H_e + b$, where b is bounded harmonic and $q(e) = 0$.

Proposition 3.6.1. *Let u and v be two pseudo-potentials on X. Then αu (for $\alpha \geq 0$) and $(u + v)$ are pseudo-potentials.*

Proof. Let $h_1 = \alpha_1 H_e + b_1$ and $h_2 = \alpha_2 H_e + b_2$ be the g.h.m. of u and v in $X \setminus e$, where b_1, b_2 are bounded harmonic. Let h be the g.h.m. of $u + v$ in $X \setminus e$. Then, $h \geq h_1 + h_2$. Now, $h \leq u + v$ implies that $h - v \leq u$. Since $h - v$ is subharmonic and h_1 is the g.h.m. of u, we conclude that $h - v \leq h_1$. Analogously, $h - h_1 \leq v$ leads to the conclusion $h - h_1 \leq h_2$. We have therefore $h = h_1 + h_2$. Similarly αh_1 is the g.h.m. of αu. Since $h = (\alpha_1 + \alpha_2)H_e + (b_1 + b_2)$ and $u(e) + v(e) = 0$, we conclude that $(u + v)$ is a pseudo-potential. Similarly, αu is a pseudo-potential. □

Proposition 3.6.2. *Let u and v be pseudo-potentials. Then,* $\inf(u, v)$ *is also a pseudo-potential.*

Proof. The g.h.m. of u and v outside a finite set are of the form $\alpha_1 H_e + b_1$ and $\alpha_2 H_e + b_2$ where $b_1$and b_2 are bounded harmonic functions. Assume without loss of generality that $\alpha_2 \leq \alpha_1$. Then, for a constant $\lambda \leq 0$, $\alpha_2 H_e + b_2 + \lambda \leq \alpha_1 H_e + b_1$ on $X \backslash e$. Hence, if h is the g.h.m. of $\inf(u, v)$ in $X \setminus e$ then $\alpha_2 H_e + b_2 + \lambda \leq h$. But $h \leq v$ on $X \backslash e$ so that $h \leq \alpha_2 H_e + b_2$ on $X \backslash e$. Consequently, $h = \alpha_2 H_e +$ (a bounded harmonic function b) in $X \backslash e$; also $[\inf(u, v)](e) = 0$. Hence $\inf(u, v)$ is a pseudo-potential with $flux_\infty[\inf(u, v)] = \alpha_2$. □

Proposition 3.6.3. *Let u be an admissible superharmonic function equivalent to a pseudo-potential. Then u is a pseudo-potential up to a constant.*

Proof. Let u be equivalent to a pseudo-potential q whose g.h.m. on $X \backslash e$ is $\alpha H_e + b$. Let h be the g.h.m. of u in $X \backslash e$. Then, $h = \alpha H_e + b +$ (a bounded harmonic function b_1) on $X \backslash e$, since u is equivalent to q. Hence, $u(x) - u(e)$ is a pseudo-potential on X. □

Remark 3.6.1. In Theorem 3.4.4, it is proved that if $e \notin E$, and if s is an admissible superharmonic function, then s and its balayage B_s^E have the same g.h.m. on $X \backslash e$. Consequently, if q is a pseudo-potential, then B_q^E also is a pseudo-potential.

Theorem 3.6.4. *Any admissible superharmonic function s in a parabolic network X is the unique sum of a pseudo-potential q and a harmonic function h.*

Proof. Let u be the g.h.m. of s in $X \backslash \{e\}$. Then, u is of the form $u = f + \alpha H_e + b$ outside a finite set. Write $h(x) = f(x) + [s(e) - f(e)]$ and $q(x) = s(x) - h(x)$, so that $h(x)$ is a harmonic function on X, $q(x)$ is a pseudo-potential and $s(x) = q(x) + h(x)$.

Suppose $s = q_1 + h_1$ is another such representation of s as the sum of a pseudo-potential and a harmonic function on X. Then, $q(x) + h(x) = q_1(x) + h_1(x)$. If $\alpha H_e + b$ is the g.h.m. of q in $X \backslash e$, with b bounded, then $\alpha H_e + b + h - h_1$ is the g.h.m. of the pseudo-potential q_1 on $X \backslash e$, so that $(b + h - h_1)$ is bounded. This implies that $(h - h_1)$ is bounded on X and hence $h - h_1 = c$, a constant. Consequently, $c = h(e) - h_1(e) = q_1(e) - q(e) = 0$. That is, $h = h_1$ and then $q = q_1$. □

Corollary 3.6.5. *(Pseudo-potentials with point harmonic support) For any y in a parabolic network X, there exists a unique pseudo-potential $Q_y(x)$ on X such that $\Delta Q_y(x) = -\delta_y(x)$.*

Proof. In Theorem 3.4.5, the existence of an admissible superharmonic function $q_y(x)$ on X such that $\Delta q_y(x) = -\delta_y(x)$ is asserted. Then, by the above theorem, $q_y(x) = Q_y(x) + h(x)$ on X where $Q_y(x)$ is a pseudo-potential and $h(x)$ is harmonic on X. Hence, $\Delta Q_y(x) = -\delta_y(x)$.

To prove the uniqueness, suppose $s(x)$ is another pseudo-potential on X such that $\Delta s(x) = -\delta_y(x)$. Write $u(x) = s(x) - Q_y(x)$. Then $\Delta u(x) = 0$, that is u is harmonic on X. Since $s(x)$ and $Q_y(x)$ are pseudo-potentials, then we have for $x \neq e$, $Q_y(x) = \alpha H_e(x) + b(x)$ and $s(x) = \beta H_e(x) + B(x)$ where $b(x)$ and $B(x)$ are bounded. Since u is harmonic on X and $u(x) = (\beta - \alpha) H_e(x) + [B(x) - b(x)]$ on $X \backslash e$, then $\beta = \alpha$ and consequently u is a constant c on X. Now $c = u(e) = s(e) - Q_y(e) = 0$. Hence, $s(x) = Q_y(x)$ for all $x \in X$. □

Theorem 3.6.6. *(Domination Principle for pseudo-potentials) Let q be a pseudo-potential on X with finite harmonic support K. Let s be a pseudo-potential up to a constant such that $flux_\infty s \geq flux_\infty q$. If $s \geq q$ on K, then $s \geq q$ on X. In particular, for a constant σ, if $q \leq \sigma$ on K, then $q \leq \sigma$ on X.*

Proof. Let $\alpha H_e + b$ be the g.h.m. of q on $X \backslash e$ and $\beta H_e + B$ be the g.h.m. of s on $X \backslash e$. Then, $\beta \geq \alpha$ by the assumption and b and B are bounded harmonic functions. Since q is harmonic outside a finite set, by Lemma 3.4.2, there exists a harmonic function h on $X \backslash e$ such that $(h - q)$ is bounded outside a finite set; hence, $h - \lambda \leq q$ on $X \backslash e$, for some constant λ. Now, $\alpha H_e + b$ being the g.h.m. of q on $X \backslash e$, we should have $h - \lambda \leq \alpha H_e + b \leq q$ on $X \backslash e$. This implies, since $(h - q)$ is bounded outside a finite set, that $q - (\alpha H_e + b) = b_1$ is bounded outside a finite set A, $e \in A$.

Write $u = s - q$ on X. Then, $\Delta u(x) \leq 0$ if $x \in X \backslash K$ and on $X \backslash A$,

$$u \geq [\beta H_e + B] - q = [\beta H_e + B] - [\alpha H_e + b + b_1] \geq B - (b + b_1),$$

since $\beta \geq \alpha$. This implies that u is lower bounded. Noting that $u \geq 0$ on K, apply the Minimum Principle (Theorem 3.4.1) for u on $X \backslash K$, to conclude that $u \geq 0$ on $X \backslash K$. Hence, $u \geq 0$ on X, that is $s \geq q$ on X.

For the particular case, take $s = \sigma$ and note that $flux_\infty s = 0$ and $flux_\infty q \leq 0$. □

Chapter 4
Schrödinger Operators and Subordinate Structures on Infinite Networks

Abstract Potential theory associated with the Schrödinger operators in a domain ω in the Euclidean space is closely related to the Laplace potential theory in ω. Based on this fact, in an infinite network X two harmonic structures, called the Laplacian and the q-Laplacian, are introduced and the interrelations between the two associated potential theories are investigated. In some sense, the q-Laplace structure is subordinate to the Laplace structure; this leads to the introduction of subordinate harmonic structures in an infinite network which already has a harmonic structure on it. This chapter deals with these subordinate structures in relation to the original structure.

On many occasions, we consider more than one harmonic structure on the same space X, and it is useful to know the interrelation between the harmonic functions defined by different structures on X. For example, if $X = (0, \infty)$, then consider the two sets of harmonic functions given by the C^2-solutions of $y'' = 0$ and $y'' = y$. In the first case, constants are harmonic and in the second case the constant 1 is a superharmonic function with e^{-x} as its g.h.m. Actually, if $u > 0$ is harmonic or superharmonic in the first harmonic structure, then $u'' = 0$ (respectively $u'' \leq 0$) so that $u'' - u \leq 0$. That is, u is superharmonic in the second harmonic structure.

As another example, in the classical potential theory in domains in the Euclidean space $\Re^n, n \geq 2$, we have the Laplace operator Δ and the Schrödinger operator Δ_q defined by $\Delta_q u(x) = \Delta u(x) - q(x)u(x)$, where usually the real-valued function $q(x)$ is assumed to be non-negative. Then there are two sets of superharmonic functions, one defined by $\Delta u \leq 0$, and the other defined by $\Delta_q u(x) \leq 0$. We can consider a similar situation in the case of discrete potential theory by comparing superharmonic functions defined by $\Delta u(x) = \sum_{y \sim x} t(x,y)[u(y) - u(x)] \leq 0$ and the q-superharmonic functions $\Delta_q u(x) = \sum_{y \sim x} t(x,y)[u(y) - u(x)] - q(x)u(x) \leq 0$. If we assume $q(x) \geq 0$ [72], then the non-negative constants are q-superharmonic functions; this fact is crucial to develop a global q-potential theory on infinite networks.

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We start with a connected infinite network $\{X, t(x,y), q(x)\}$ with a countable number of vertices, $\{t(x,y)\}$ being the set of conductance not necessarily symmetric, and $q(x)$ is a real-valued function on X. We suppose that X is locally finite, that is every vertex has only a finite number of vertices as neighbours. If we do not presuppose that X is locally finite, then we may have to consider only functions $u(x)$ which satisfy the condition that for any vertex x, $\sum_{y \sim x} t(x,y)u(y)$ is absolutely convergent; also, we may have to assume that $t(x) = \sum_{y \sim x} t(x,y)$ is finite for every vertex x. We define, without placing any restriction on the sign of the function $q(x)$, $\Delta_q u(x) = \Delta u(x) - q(x)u(x)$ and say that u defined on a subset F of X is q-superharmonic if $\Delta_q u(x) \leq 0$ for every $x \in \overset{0}{F}$. Remark that given any function $\xi > 0$, if we take $q(x) = \frac{\Delta\xi(x)}{\xi(x)}$, then $\xi(x)$ is a positive q-harmonic function on X. However, in the general case, $\Delta_q u \leq 0$ may not have a positive solution. For example, at some vertex $x_0 \in X$, if we have $t(x_0) + q(x_0) < 0$, then there can be no positive q-superharmonic function on X.

4.1 Local Properties of q-Superharmonic Functions

In this section, $q(x)$ is an arbitrary real-valued function on X; recall that a real-valued function u defined on X is said to be q-superharmonic on X if $\Delta u(x) \leq q(x)u(x)$ for every $x \in X$. The following properties are readily proved:

1. If u and v are q-superharmonic on X, and if α, β are two non-negative numbers, then $\alpha u + \beta v$ and $\inf(u,v)$ are q-superharmonic functions on X.
2. If u_n is a sequence of q-superharmonic functions and if $u(x) = \lim u_n(x)$ exists and is finite for each x, then u is q-superharmonic on X.
3. For any finite set F, there exists a constant $M > 0$ such that for any real-valued function f on X, $|\Delta_q f(x)| \leq M \sum_{y \in F} |f(y)|$ for any $x \in \overset{0}{F}$.

Proof. Let $x \in \overset{0}{F}$. For any f on X,

$$|\Delta_q f(x)| = \left| \sum_{y \sim x} t(x,y)f(y) - [t(x) + q(x)]\, f(x) \right|$$

$$= \left| \sum_{y \in F} t(x,y)f(y) - [t(x) + q(x)]\, f(x) \right|,$$

(since $x \in \overset{0}{F}$, if $y \sim x$, then $y \in F$.)

$$\leq \sum_{y \in F} t(x,y)\, |f(y)| + [t(x) + |q(x)|]\, |f(x)|$$

$$\leq [t(x) + |q(x)|] \left[|f(x)| + \sum_{y \in F} |f(y)| \right]$$
$$\leq M \sum_{y \in F} |f(y)|, \text{ where } M = 2 \max_{x \in \overset{0}{F}} [t(x) + |q(x)|].$$

4. Let $u \geq 0$ be q-superharmonic on X. Suppose $u(x) = 0$ for some x in X, then $u \equiv 0$.
   Proof. $0 = [t(x) + q(x)]\, u(x) \geq \sum_{y \sim x} t(x, y) u(y) \geq 0$. Hence, $u(y) = 0$ if $y \sim x$. Then, by the connectedness of X, it follows that $u(z) = 0$ for all z in X.
5. Let $t(x_0) + q(x_0) \leq 0$ at some vertex $x_0 \in X$. Then there is no positive q-superharmonic function on X.
   Proof. Suppose $u \geq 0$ is q-superharmonic on X. Then,
   $$0 \geq [t(x_0) + q(x_0)] u(x_0) \geq \sum_{y \sim x_0} t(x_0, y) u(y) \geq 0,$$
   which implies that $u(y) = 0$ at any $y \sim x_0$. Hence, by the above minimum principle, $u \equiv 0$.
6. There exists a positive q-superharmonic function on X if and only if $q(x) \geq \frac{\Delta \xi(x)}{\xi(x)}$ on X, for some $\xi > 0$ defined on X.
   Proof. Suppose the condition is satisfied. Then $\xi > 0$ is a q-superharmonic function on X. For, $\Delta_q \xi(x) = \sum_{y \sim x} t(x, y)[\xi(y) - \xi(x)] - q(x)\xi(x) \leq \Delta \xi(x) - \Delta \xi(x) = 0$. Conversely, suppose there exists a q-superharmonic function $s > 0$ on X; that is, $\Delta s(x) - q(x)s(x) = \Delta_q s(x) \leq 0$. Then, $q(x) \geq \frac{\Delta s(x)}{s(x)}$ on X.
7. (Harnack property for q-superharmonic functions) Let A and B be two disjoint finite subsets of X. Assume $t(x) + q(x) > 0$ for every vertex x in X. Then there exists a constant $\alpha > 0$ such that $u(x) \geq \alpha u(y)$ if $x \in A$, $y \in B$, for any non-negative q-superharmonic function u on X.
   Proof. Since A and B are finite sets, it is enough to prove that if $x \neq y$, then there exists a constant $\alpha > 0$ such that $u(x) \geq \alpha u(y)$ for any q-superharmonic function $u \geq 0$ on X. Choose a path $\{x = x_0, x_1, ..., x_n = y\}$ connecting x and y. Let $u > 0$ be a q-superharmonic function on X. Then, $[t(x) + q(x)]u(x) \geq \sum_{y \sim x} t(x, y)u(y) \geq t(x, x_1)u(x_1)$. Similarly, $[t(x_1) + q(x_1)]u(x_1) \geq \sum_{y \sim x_1} t(x_1, y)u(y) \geq t(x_1, x_2)u(x_2)$. Consequently, $u(x) \geq \frac{t(x,x_1)}{[t(x)+q(x)]} \times \frac{t(x_1,x_2)}{[t(x_1)+q(x_1)]} u(x_2)$. Proceeding similarly, we find that if $\alpha = \frac{t(x,x_1)...t(x_{n-1},y)}{[t(x)+q(x)]...[t(x_{n-1})+q(x_{n-1})]}$, then $\alpha > 0$ and $u(x) \geq \alpha u(y)$.

Theorem 4.1.1. *Assume that* $[t(x) + q(x)] > 0$ *for every* $x \in X$. *Let* F *be a subset of* X. *Suppose* u *is a* q*-superharmonic function on* F *and* v *is a* q*-subharmonic function on* F *such that* $u \geq v$. *Then there exists a* q*-harmonic function* h *on* F *such that* $u \geq h \geq v$. *This function* h *can be chosen such that if* h' *is another* q*-harmonic function and if* $u \geq h' \geq v$ *on* F, *then* $h \geq h'$ *on* F.

Proof. Let $\Im$ be the family of all q-subharmonic functions g on F such that $u \geq g \geq v$ on F. Let $a \in \overset{0}{F}$. Then, $\sum_{y \sim a} t(a,y)g(y) \geq [t(a) + q(a)]g(a)$. Consider the function $g^{'}$ defined on F such that $g^{'}(y) = g(y)$ if $y \neq a$ and $g^{'}(a) = \frac{1}{[t(a)+q(a)]} \sum_{y \sim a} t(a,y)g(y)$. The function $g^{'}(x)$ is usually denoted by $P_a g(x)$, and is called the Δ_q-Poisson modification of g at the vertex a. Then, $g^{'} \geq g$ on F, and

$$\Delta_q g^{'}(a) = \sum_{y \sim a} t(a,y)[g^{'}(y) - g^{'}(a)] - q(a)g^{'}(a)$$

$$= \sum_{y \sim a} t(a,y)g(y) - [t(a) + q(a)]g^{'}(a) = 0.$$

Hence, $g^{'}$ is q-harmonic at the vertex a.

If $z \in \overset{0}{F}$ and $z \neq a$, then

$$\Delta_q g^{'}(z) = \sum_{y \sim z} t(z,y)g^{'}(y) - [t(z) + q(z)]g^{'}(z)$$

$$\geq \sum_{y \sim z} t(z,y)g(y) - [t(z) + q(z)]g(z),$$

since $g^{'} \geq g$ on F and $g^{'}(z) = g(z)$ if $z \neq a$. Consequently, $\Delta_q g^{'}(z) \geq \Delta_q g(z) \geq 0$. Thus, $g^{'}$ is q-subharmonic on F and q-harmonic at the vertex a. Moreover, $g^{'}(a) \leq \frac{1}{[t(a)+q(a)]} \sum_{y \sim a} t(a,y)u(y) \leq u(a)$, since u is q-superharmonic. Consequently, $g^{'} \leq u$ on F. Hence, $g^{'} \in \Im$, so that $\Im$ is a Perron family of q-subharmonic functions on F (Sect. 2.4).

Let $h(x) = \sup_{g \in \Im} g(x)$ for every $x \in F$. Since each $g \leq u$, we find $h \leq u$ on F. Let $a \in \overset{0}{F}$, and let $V(a)$ denote the set of neighbours of a, together with a. Then, $V(a)$ is a finite set so that we can find a sequence $\{g_x^{(n)}\}$ in $\Im$ for every $x \in V(a)$ such that $g_x^{(n)} \to h(x)$ as $n \to \infty$. Since $\Im$ is an upper directed family of functions, and since $V(a)$ contains only finitely many vertices, there exists u_n in $\Im$ such that $u_n \geq g_x^{(n)}$ for every $x \in V(a)$. Hence $u_n(x) \to h(x)$ for every $x \in V(a)$. Let $v_n = P_a u_n$. Then, as shown above, $v_n \in \Im$ and $v_n(x) \to h(x)$ as $n \to \infty$ for every $x \in V(a)$ and v_n is q-harmonic at a. Consequently, we have $\Delta_q h(a) = \lim_{n \to \infty} \Delta_q v_n(a) = 0$. That is, h is q-harmonic at a. Thus, h is a q-harmonic function on F such that $u \geq h \geq v$ on F.

Suppose $h^{'}$ is a q-harmonic function on F such that $u \geq h^{'} \geq v$. Then $h^{'} \in \Im$, so that $h^{'} \leq h$ on F. □

Remark 4.1.1. We say that a non-negative q-superharmonic function p on a set $E \subset X$ is a q-potential on E if for every q-subharmonic function v on E such that $v \leq p$ on E, we have $v \leq 0$. Thus, with the notations of the above theorem, $p = u - h$ is a q-potential on F and $u = p + h$ is the unique representation of u as the sum of a q-potential and a q-harmonic function on F.

Let s be a q-superharmonic function on X. Suppose A is the smallest subset of X such that s is q-harmonic at each vertex of $X\backslash A$. Then A is referred to as the *q-harmonic support* of s in X.

Theorem 4.1.2. *(q-Domination Principle) Let $[q(x) + t(x)] > 0$ for every x in X. Let p be a q-potential with q-harmonic support A. Let $s > 0$ be any q-superharmonic function on X such that $s \geq p$ on A. Then $s \geq p$ on X.*

Proof. Let $u = \inf(s, p)$. Then, u is q-superharmonic on X and $u \leq p$ on X and $u = p$ on A. Write $v = p - u$ on X. Then, for $a \in A$,

$$\Delta_q v(a) = \sum_{y\sim a} t(a,y)v(y) - [q(a) + t(a)]v(a) = \sum_{y\sim a} t(a,y)v(y) \geq 0.$$

For $x \in X\backslash A$, $\Delta_q v(x) = \Delta_q p(x) - \Delta_q u(x) = 0 - \Delta_q u(x) \geq 0$. Hence, v is q-subharmonic on X and $v \leq p$ on X. Hence $v \leq 0$ on X, so that $p \leq u$; but $u \leq p$. Consequently, $s \geq p$ on X. □

Theorem 4.1.3. *Assume that $[t(x) + q(x)] > 0$ for every $x \in X$. Let $f(x)$ be a real-valued function on X. Suppose the family $\Im$ of q-superharmonic functions s majorizing f on X is non-empty. Then $Rf(x) = \inf_{s\in\Im} s(x)$ is q-superharmonic on X and q-harmonic at each vertex a where $f(x)$ is q-subharmonic.*

Proof. $\Im$ is a lower-directed family of q-superharmonic functions. Hence Rf is a q-superharmonic function on X. For, if x and y are two vertices in X, choose two decreasing subsequences $\{u_n\}$ and $\{v_n\}$ such that $Rf(x) = \lim u_n(x)$ and $Rf(y) = \lim v_n(y)$. Since $\Im$ is lower-directed, we can choose a decreasing sequence $\{w_n\}$ from $\Im$ such that $Rf(x) = \lim w_n(x)$ and $Rf(y) = \lim w_n(y)$. Since X is a countable set of vertices, this method will produce a decreasing subsequence $\{s_n\}$ from $\Im$ such that $Rf(z) = \lim s_n(z)$ for every $z \in X$. Since $Rf(z) \geq f(z)$, we conclude that Rf is a q-superharmonic function on X.

Suppose $f(x)$ is q-subharmonic at $x = a$. Let $s \in \Im$. Consider the function $s_a(x)$ on X such that $s_a(x) = s(x)$ if $x \neq a$ and $s_a(a) = \sum_{y\sim a} \frac{t(a,y)}{t(a)+q(a)} s(y) \leq s(a)$. Then, as shown in the proof of the above theorem, $s_a(x)$ is q-superharmonic on X and q-harmonic at $x = a$. Moreover, at $x = a$ where $f(x)$ is q-subharmonic, we have

$$s_a(a) = \sum_{y\sim a} \frac{t(a,y)}{t(a) + q(a)} s(y) \geq \sum_{y\sim a} \frac{t(a,y)}{t(a) + q(a)} f(y) \geq f(a).$$

Hence, $s_a(x) \geq f(x)$ for all $x \in X$. Consequently, $s_a \in \Im$. This implies that $Rf(x) = \inf_{s\in\Im} s(x)$ is q-harmonic at $x = a$. □

Corollary 4.1.4. *(q-Balayage) Let $s > 0$ be a q-superharmonic function on X and let A be an arbitrary subset of X. Then there exists a q-superharmonic function $R_s^A > 0$ on X such that $R_s^A \leq s$ on X, $R_s^A = s$ on A and R_s^A is q-harmonic at each vertex in $X \backslash A$.*

Proof. Since $s > 0$ is q-superharmonic on X, for any $x \in X$,

$$t(x) + q(x) \geq \sum_{y \sim x} t(x,y) \frac{s(y)}{s(x)} > 0.$$

Define $f \geq 0$ on X such that $f = s$ on A and $f = 0$ on $X \backslash A$. Note that for any $x \in X \backslash A$, we have $\Delta_q f(x) = \sum_{y \sim x} t(x,y) f(y) \geq 0$, that is f is q-subharmonic at each vertex in $X \backslash A$.

Now, if we set $R_s^A = Rf$ on X, then by the above theorem $R_s^A > 0$ is q-superharmonic on X and q-harmonic at each vertex of $X \backslash A$. Since the q-superharmonic function $s \geq f$ on X, we conclude that $R_s^A \leq s$ on X; but $R_s^A \geq f = s$ on A so that $R_s^A = s$ on A. □

Note. 1. The q-superharmonic function R_s^A is the smallest in the following sense: If u is a non-negative function on X such that $u \leq s$ on X, $u = s$ on A and u is q-harmonic at each vertex of $X \backslash A$, then $u \geq R_s^A$ on X.

For, if $a \in A$, then since $u(x) \leq s(x)$ for all X and since $u(a) = s(a)$, we have

$$\begin{aligned} \Delta_q u(a) &= \sum_{y \sim a} t(a,y) u(y) - [t(a) + q(a)] u(a) \\ &\leq \sum_{y \sim a} t(a,y) s(y) - [t(a) + q(a)] s(a) \\ &\leq 0. \end{aligned}$$

By hypothesis, $\Delta_q u(x) = 0$ if $x \in X \backslash A$. Hence $\Delta_q u \leq 0$ on X. That is, u is q-superharmonic on X and $u = s$ on A. Hence, $u \geq R_s^A$ (Corollary 4.1.4).

2. Suppose there exist q-potentials on X. Then, for any vertex e, there exists a unique q-potential $G_e(x) > 0$ X such that $\Delta_q G_e(x) = -\delta_e(x)$, for all $x \in X$.

For, if $p > 0$ is a q-potential on X, take $s(x) = p(x)$ and $A = \{e\}$ in the above corollary and write

$$G_e(x) = \frac{R_p^{\{e\}}(x)}{(-\Delta_q) R_p^{\{e\}}(e)};$$

we know that a positive q-superharmonic function dominated by a q-potential is itself a q-potential. Suppose $Q > 0$ is another q-potential such that $\Delta_q Q(x) = -\delta_e(x)$. Then, $Q(x) = G_e(x)$+[a q-harmonic function $h(x)$] in X; and, by the uniqueness of decomposition of a positive q-superharmonic function as the sum of a q-potential and a q-harmonic function, we note that $h \equiv 0$. (see Theorem 4.1.11 below for a generalization of this result.)

Since the potential theoretic study of functions on X is fertile if there is at least one positive q-superharmonic function on X, we shall *assume in the sequel that there exists a function* $\xi > 0$ *on X, such that* $q(x) \geq \frac{\Delta\xi(x)}{\xi(x)}$ *on X.* As a consequence, we have $[t(x) + q(x)] > 0$ for every vertex x in X. For,

$$q(x) \geq \frac{\Delta\xi(x)}{\xi(x)} = \sum_{y\sim x} t(x,y)\frac{[\xi(y) - \xi(x)]}{\xi(x)} = \left[\sum_{y\sim x} t(x,y)\frac{\xi(y)}{\xi(x)}\right] - t(x).$$

Hence, $[t(x) + q(x)] \geq \sum_{y\sim x} t(x,y)\frac{\xi(y)}{\xi(x)} > 0$. Then the following lemma (which is basically in Bendito et al. [20]) is useful.

Lemma 4.1.5. *Let* u *be a real-valued function defined on* X *and let* $v = \frac{u}{\xi}$. *Then,* $\Delta_q u(x) = \sum_{y\sim x} t(x,y)\xi(y)[v(y) - v(x)] + [\Delta\xi(x) - \xi(x)q(x)]v(x)$.

Proof.

$$\begin{aligned} \Delta_q u(x) &= \Delta u(x) - q(x)u(x) \\ &= [\Delta u(x) - \frac{\Delta\xi(x)}{\xi(x)}u(x)] + [\frac{\Delta\xi(x)}{\xi(x)} - q(x)]u(x). \\ &= \frac{1}{\xi(x)}[\xi(x)\Delta u(x) - u(x)\Delta\xi(x)] + [\Delta\xi(x) - \xi(x)q(x)]v(x). \end{aligned}$$

Now,

$$\begin{aligned} \xi(x)\Delta u(x) - u(x)\Delta\xi(x) &= \xi(x)[\sum_{y\sim x} t(x,y)\{u(y) - u(x)\}] \\ &\quad -u(x)[\sum_{y\sim x} t(x,y)\{\xi(y) - \xi(x)\}] \\ &= \sum_{y\sim x} t(x,y)[\xi(x)u(y) - \xi(y)u(x)] \\ &= \sum_{y\sim x} t(x,y)\xi(x)\xi(y)\left[\frac{u(y)}{\xi(y)} - \frac{u(x)}{\xi(x)}\right] \\ &= \xi(x)\sum_{y\sim x} t(x,y)\xi(y)[v(y) - v(x)]. \end{aligned}$$

The lemma is proved. □

Theorem 4.1.6. *(Minimum Principle for the q-Laplacian) Let* F *be a finite subset of* X. *Let* u *be a* q*-superharmonic function on* F *such that* $u \geq 0$ *on* ∂F. *Then* $u \geq 0$ *on* F.

Proof. Let $v = \frac{u}{\xi}$. Suppose u takes negative values in F. Then, v takes negative values on F. Let $-\lambda = \min_{x \in F} v(x)$. Then, for some $z \in \overset{0}{F}$, $v(z) = -\lambda$. Note $u(z) < 0$. Then, by the above Lemma 4.1.5,

$$0 \geq \Delta_q u(z) = \sum_{y \sim z} t(z,y)\xi(y)[v(y) - v(z)] + [\Delta\xi(z) - \xi(z)q(z)]v(z)$$

$$= \text{[a non-negative term] + [a non-positive term][a negative term]} \geq 0.$$

We conclude, in particular, that $v(y) = v(z) = -\lambda$ if $y \sim z$.

Let $a \notin F$. Since X is connected, there is a path $\{z, y_1, y_2, ..., y_n = a\}$ connecting z and a. Take the smallest index i such that $y_i \notin \overset{0}{F}$ and $y_{i-1} \in \overset{0.}{F}$. Then, $y_i \in \partial F$ and hence $u(y_i) \geq 0$ so that $v(y_i) \geq 0$. But $v(y_i) = v(y_{i-1}) = ... = v(z) = -\lambda$, a contradiction. Hence, v does not take negative values on F. Since $v \geq 0$, we conclude that $u \geq 0$ on F. □

Remark 4.1.2. The above theorem in the case of a finite network when $q(x) \geq 0$ and $q(x_0) > 0$ for at least one vertex x_0 has been proved as Lemma 2.4.3.

Theorem 4.1.7. *(Generalised Dirichlet solution for the q-Laplacian) Let F be a subset of X. Let E be a subset in $\overset{0}{F}$. Let f be a real-valued function on $F \backslash E$. Suppose there exist functions u and v on F such that $\Delta_q u \leq 0$ and $\Delta_q v \geq 0$ on E, $u \geq v$ on F and $u \geq f \geq v$ on $F \backslash E$. Then, there exists a function h on F such that $h = f$ on $F \backslash E$, $u \geq h \geq v$ on F and $\Delta_q h = 0$ at every vertex in E.*

Proof. Let u_1 be the function on F such that $u_1 = f$ on $F \backslash E$ and $u_1 = u$ on E. Then, u_1 is q-superharmonic at every vertex in E. For, let x be a vertex in E. If x and all its neighbours are in E, then $\Delta_q u_1(x) = \Delta_q u(x) \leq 0$; on the other hand if x has one or more neighbours in $F \backslash E$, then

$$\begin{aligned}\Delta_q u_1(x) &= \Delta u_1(x) - q(x)u_1(x)\\ &\leq \Delta u(x) - q(x)u(x), \text{since } u(z) \geq f(z) = u_1(z) \text{ if } z \in F\backslash E\\ &\quad \text{and } u(x) = u_1(x)\\ &= \Delta_q u(x) \leq 0.\end{aligned}$$

Similarly, if v_1 is the function on F such that $v_1 = f$ on $F \backslash E$ and $v_1 = v$ on E, then v_1 is q-subharmonic at every vertex in E. Moreover, $u_1 \geq v_1$ on F, so that as in the proof of Theorem 4.1.1 we can construct a function h on F which is q-harmonic at every vertex of E and $u_1 \geq h \geq v_1$ on F. Moreover, $h = f$ on ∂F; and this function h is the largest one by construction. □

Corollary 4.1.8. *(Classical Dirichlet solution for the q-Laplacian) Let F be a finite subset of X. Let f be a real-valued function defined on ∂F. Then, there exists a unique q-harmonic function h on F such that $h = f$ on ∂F.*

Proof. Since $\xi > 0$ is q-superharmonic on X, we can find a constant $\lambda > 0$ such that $-\lambda\xi(x) \leq f(x) \leq \lambda\xi(x)$ if $x \in \partial F$. Then, by Theorem 4.1.7, there exists a function h on F such that $h = f$ on ∂F and $\Delta_q h(x) = 0$ if $x \in \overset{0}{F}$. The uniqueness of h follows from the Minimum Principle given in Theorem 4.1.6. □

We have assumed that there exists a function $\xi > 0$ on X such that $q(x) \geq \frac{\Delta\xi(x)}{\xi(x)}$. If it so happens that $q(x) = \frac{\Delta\xi(x)}{\xi(x)}$, then $\xi(x)$ is a positive q-harmonic function on X. Even if it is not the case, the following theorem shows that there always exists at least one positive q-harmonic function on X.

Theorem 4.1.9. *There exists a q-harmonic function $h > 0$ on X.*

Proof. Fix a vertex z. Let $\{K_n\}_{n\geq 1}$ be an increasing sequence of finite sets such that $z \in \overset{0}{K_n} \subset K_n \subset \overset{0}{K}_{n+1} \subset K_{n+1}$ and $X = \cup K_n$. Let s_n be the function on X defined as the Dirichlet solution in K_n with boundary values $\xi(x)$ and extended by $\xi(x)$ outside K_n. Write $h_n(x) = \frac{s_n(x)}{s_n(z)}$. Then, h_n is q-superharmonic on X, $\Delta_q h_n(x) = 0$ for $x \in \overset{0}{K_n}$ and $h_n(z) = 1$.

We know that (Property 7, Sect. 4.1) for any $y \in X$, there exists a constant $\alpha(y) > 0$ such that $u(y) \leq \alpha(y)u(z)$ for any q-superharmonic function $u > 0$ on X. In particular, for any $x \in X$, $h_n(x) \leq \alpha(x)h_n(z) = \alpha(x)$, so that $\{h_n(x)\}$ is a bounded sequence of real numbers. Since X is a countable set, we can extract a subsequence $\left\{h_n^{'}\right\}$ from $\{h_n\}$ such that $h(x) = \lim_{n\to\infty} h_n^{'}(x)$ exists for each x in X. Now, given any finite set F in X, we can find an integer m such that $h_n^{'}$ is q-harmonic at each vertex of F if $n \geq m$. Consequently, h is q-harmonic at each vertex of F. The set F being an arbitrary finite set, we conclude that $h(x)$ is a non-negative q-harmonic function on X. Since $h(z) = 1$, by the Minimum Principle (Property 4, Sect. 4.1), $h > 0$ on X. This proves the existence of a positive q-harmonic function on X. □

The following theorem is a generalisation of the well-known *condenser principle* in finite electrical networks, for a condenser with positive and negative plates.

Theorem 4.1.10. *Let A and B be two arbitrary disjoint subsets of X. Then, there exists u on X such that $0 \leq u(x) \leq \xi(x)$ for $x \in X$, $u = \xi$ on A, and $u = 0$ on B. Moreover, $\Delta_q u = 0$ on $X\backslash(A \cup B)$, $\Delta_q u \leq 0$ on A, and $\Delta_q u \geq 0$ on B.*

Proof. Let $E = X\backslash(A \cup B)$. Let $F = V(E)$ be the set consisting of E and all the neighbours of each vertex in E; then, $E \subset \overset{0}{F}$. Let f be the function defined on $F\backslash E$ such that $f = \xi$ on $(F\backslash E) \cap A$ and $f = 0$ on $(F\backslash E) \cap B$. Then, $0 \leq f \leq \xi$ on $F\backslash E$. Hence, by Theorem 4.1.7, there exists a function u on F such that $u = f$ on $F\backslash E$, $0 \leq u \leq \xi$ on F and $\Delta_q u = 0$ at every vertex on E. Extend u by ξ on A and by 0 on B. Then, u is a function defined on X.

If $a \in A$, then

$$\begin{aligned}\Delta_q u(a) &= \sum_{y\sim a} t(a,y)[u(y)-\xi(a)] - q(a)\xi(a), \text{ since } u(a)=\xi(a)\\ &\le \sum_{y\sim a} t(a,y)[\xi(y)-\xi(a)] - q(a)\xi(a) = \Delta_q \xi(a) \le 0.\end{aligned}$$

If $b \in B$, then

$$\begin{aligned}\Delta_q u(b) &= \sum t(b,y)[u(y)-0] - 0, \text{ since } u(b)=0\\ &\ge 0.\end{aligned}$$

□

The following result is a crucial one which asserts the existence of q-superharmonic functions on X with point q-harmonic support, like the Green function and the Logarithmic function in the classical potential theory on the Euclidean spaces.

Theorem 4.1.11. *For any vertex e in X, there exists a q-superharmonic function s on X such that $\Delta_q s(x) = -\delta_e(x)$ for every $x \in X$.*

Proof. Let $\{E_n\}$ be a sequence of finite connected, circled sets such that $e \in \overset{\circ}{E}_1$, $E_n \subset \overset{0}{E}_{n+1}$ for $n \ge 1$, and $X = \cup E_n$. Let $x_1 \sim e$ and let E be the set of vertices consisting of all y such that a path connecting y to e passes through x_1. Then E is connected and every vertex in E other than e is in $\overset{0}{E}$. Since X has an infinite number of vertices, we can choose E as an infinite set. Let $F_n = E \cap E_n$ and let h_n be the q-Laplacian Dirichlet solution in F_n with boundary value 0 at e and $\xi(x)$ at other boundary vertices x of F_n. Let $u_n(x) = \frac{h_n(x)}{h_n(x_1)}$. Then, as explained earlier, we can extract a convergent subsequence $\left\{u_n^{'}\right\}$ of $\{u_n\}$ such that $u_n^{'}(x)$ tends to a finite limit $u(x)$ at every vertex in E.

Now, given any vertex $a \in E\setminus\{e\}$, $u_n^{'}(x)$ is q-harmonic at $x = a$ for n large. Hence, $u(x)$ is q-harmonic on E such that $u(e) = 0$, $u(x_1) = 1$. Define a function $v \ge 0$ on X such that $v = u$ on E and $v = 0$ outside E. Then, $v \ge 0$ is q-subharmonic on X and q-harmonic on $X\setminus\{e\}$. Remark that $\Delta_q v(e) \ne 0$; for, if $\Delta_q v(e) = 0$, then $v \ge 0$ is q-harmonic on X and takes the value 0 at e. Consequently, we arrive at the contradiction that $v \equiv 0$. Hence, $\Delta_q v(e) \ne 0$. Now, set $s(x) = \frac{-v(x)}{\Delta_q v(e)}$, for $x \in X$. □

Corollary 4.1.12. *(q-Green's function for a finite set) Let F be a finite set in X. Let $e \in \overset{0}{F}$. Then, there exists a unique q-superharmonic function $G_e^F(x) \ge 0$ on F such that $\Delta_q G_e^F(x) = -\delta_e(x)$ for $x \in \overset{0}{F}$ and $G_e^F(z) = 0$ for each $z \in \partial F$.*

Proof. Let $s(x)$ be a q-superharmonic function on X such that $\Delta_q s(x) = -\delta_e(x)$ for each x in X. Let $h(x)$ be the Dirichlet q-Laplacian solution for F with boundary values $s(x)$ on ∂F. Then, $G_e^F(x) = s(x) - h(x)$ defined on F has the properties stated in the theorem. The uniqueness of $G_e^F(x)$ follows from the Minimum Principle Theorem 4.1.6. □

Theorem 4.1.13. *(q-Equilibrium Principle) Let $f(x)$ and $g(x)$ be two real-valued functions defined on X. Let E be a finite subset of X. Then there exists a unique function u on X such that $\Delta_q u(a) = f(a)$ for each a in E and $u(z) = g(z)$ for each z in $X \backslash E$.*

Proof. Let $F = V(E)$. Then, $E \subset \overset{0}{F}$. For $a \in E$, let s be a q-superharmonic function on X such that $\Delta_q s(x) = -\delta_a(x)$ in X. Choose $\lambda > 0$ large so that $-\lambda\xi(y) \leq s(y)$ for every $y \in F$. Then, by Theorem 4.1.7, there exists a function h on F such that $\Delta_q h(x) = 0$ if $x \in E$ and $h(x) = s(x)$ if $x \in F \backslash E$. Define $g_a(x)$ on X such that $g_a(x) = s(x) - h(x)$ on F extended by 0 outside F. Then, $\Delta_q g_a(x) = -\delta_a(x)$ for every $x \in E$ and $g_a(z) = 0$ for each $z \in X \backslash E$. Write $v(x) = -\sum_{a \in E} f(a) g_a(x)$, for x in X. Then, $\Delta_q v(a) = f(a)$ for each a in E, and $v(z) = 0$ if $z \in X \backslash E$.

Again, choose some $\alpha > 0$, so that $-\alpha\xi(y) \leq g(y) \leq \alpha\xi(y)$ for $y \in F$. Then, there exists $h_1(y)$ on F such that $\Delta_q h_1(x) = 0$ if $x \in E$ and $h_1(x) = g(x)$ for $x \in F \backslash E$. Let $w(x)$ be the function defined on X such that $w(y) = h_1(y)$ if $y \in F$ and $w(y) = g(y)$ if $y \in X \backslash F$. Then, $\Delta_q w(a) = 0$ if $a \in E$ and $w(z) = g(z)$ if $z \in X \backslash E$.

Write $u = v + w$ in X. Then, $\Delta_q u(a) = f(a)$ if $a \in E$ and $u(z) = g(z)$ if $z \in X \backslash E$. For the uniqueness of the function u, we rely on the result (proved as in Theorem 4.1.6) that if E is a finite set in X and if $k(x)$ is a function on X such that $\Delta_q k(a) = 0$ if $a \in E$ and $k(z) = 0$ if $z \in X \backslash E$, then $k \equiv 0$. □

4.2 Classification of q-Harmonic Networks

We continue with our assumption that there exists a function $\xi > 0$ on the infinite network X such that $q(x) \geq \frac{\Delta\xi(x)}{\xi(x)}$ for every $x \in X$. However, now we shall make a simplification. By Theorem 4.1.9, there exists a q-harmonic function $h > 0$ on X, that is $h(x)q(x) = \Delta h(x)$ for every $x \in X$. Hence, without loss of generality, *we shall assume henceforth that $q(x) = \frac{\Delta\xi(x)}{\xi(x)}$ for $x \in X$*. But the existence of a q-harmonic function on X does not insure the existence of q-potentials on X. We give now a necessary and sufficient condition for the existence of q-potentials on X by using a property of "the point at infinity".

ξ-harmonic measure of the point at infinity: Let e be a fixed vertex. Let $\{E_n\}$ be a sequence of finite connected circled sets such that $e \in \overset{\circ}{E}_1$, $E_n \subset E_{n+1}$ for $n \geq 1$,

and $X = \bigcup_{n \geq 1} E_n$. Take $F = E_n$, $E = \overset{0}{F} \setminus \{e\}$. Then, by Theorem 4.1.7, there exists a unique q-harmonic function u_n on F such that $u_n(e) = 0$, $u_n(z) = \xi(z)$ if $z \in \partial F$, and $\Delta_q u_n(x) = 0$ if $x \in E$. Assume that u_n has been defined on X by taking $u_n(z) = \xi(z)$ if $z \in X \setminus F$. Then, $\{u_n\}$ is a decreasing sequence of q-superharmonic functions on $X \setminus \{e\}$, such that $\Delta_q u_n(x) = 0$ if $x \in \overset{\circ}{E}_n \setminus \{e\}$. Let $h(x) = \lim_{n \to \infty} u_n(x)$ for $x \in X$. Then, $0 \leq h(x) \leq \xi(x)$ for every $x \in X$, $h(e) = 0$, $\Delta_q h(e) \geq 0$, and $\Delta_q h(x) = 0$ if $x \neq e$. We refer to $h(x)$ as the Dirichlet solution with boundary values 0 at e and ξ at the point infinity. It is possible that $h \equiv 0$ on X in which case we say that the *ξ-harmonic measure of the point at infinity is 0*; otherwise, the *ξ-harmonic measure of the point at infinity is positive*.

Theorem 4.2.1. *The ξ-harmonic measure of the point at infinity is positive if and only if there exist q-potentials on X.*

Proof. Suppose the ξ-harmonic measure of the point at infinity is positive. Then, $\xi(x) - h(x)$ is a positive q-superharmonic function that is not q-harmonic on X. For, if h is q-harmonic on X, then by the Minimum Principle (Property 4, Sect. 4.1) $h \equiv 0$ on X. Hence, by the remark following Theorem 4.1.1, $\xi(x) - h(x)$ is the sum of a positive q-potential and a non-negative q-harmonic function on X. This proves the existence of a positive q-potential on X.

Conversely, suppose there exists a q-potential on X. Then, by Note 2 following Corollary 4.1.4 we can construct a q-potential p on X with q-harmonic support $\{e\}$. Then, $u(x) = \frac{\xi(e)}{p(e)} p(x)$ is a q-potential with q-harmonic support at $\{e\}$ and $u(e) = \xi(e)$. Hence, by the Domination Principle given in Theorem 4.1.2, $u(x) \leq \xi(x)$ for every $x \in X$. Let $v(x) = \xi(x) - u(x)$. Then on E_n, $0 \leq v(x) \leq u_n(x)$ where $u_n(x)$ is the above-defined q-Dirichlet solution in E_n with $\xi(x)$on ∂E_n and 0 at e. Taking limits when $n \to \infty$, we see that $v(x) \leq h(x)$; that is, $0 \leq \xi(x) - u(x) \leq h(x)$ on X. Since $\xi(x)$ is q-harmonic on X and $u(x)$ is a positive q-potential on X, the function ξ is different from u. Hence, h is not identically 0. That is, the $\xi-$harmonic measure of the point at infinity is positive. □

Now, write $v = \frac{u}{\xi}$ for any real-valued function u on X. Then, by Lemma 4.1.5,

$$\Delta_q u(x) = \sum_{y \sim x} t(x,y)\xi(y)[v(y) - v(x)] - [\xi(x)q(x) - \Delta\xi(x)]v(x).$$

Let us consider a new set of conductance $t^{\xi}(x,y) = t(x,y)\xi(y)$ defined on the edge $[x,y]$. With this set of conductance, let us denote the Laplacian on X by $\Delta^{\xi} v(x) = \sum_{y \sim x} t(x,y)\xi(y)[v(y) - v(x)]$.

Then, $\Delta_q u(x) = \Delta^{\xi} v(x)$ for $x \in X$, since $q(x) = \frac{\Delta\xi(x)}{\xi(x)}$. Consequently, u is q-harmonic (respectively, q-superharmonic) if and only if $v = \frac{u}{\xi}$ is Δ^{ξ}-harmonic (respectively, Δ^{ξ}-superharmonic) on X. Thus, a potential-theoretic problem in X with the original set of conductance $\{t(x,y)\}$ can be stated as a problem in X with the new set of conductance $\left\{t^{\xi}(x,y)\right\}$ in which, by the assumption, the constant

1 is Δ^{ξ}-harmonic. The solution in the new system then can be transferred back to the original system. In particular, the classification of the network $\left\{X, t^{\xi}(x,y)\right\}$ by using the Laplacian Δ^{ξ} follows the pattern described in Chap. 3. Thus, the problems connected with the q-harmonic classification of the original network $\{X, t(x,y), q(x)\}$ can be solved in the associated network $\left\{X, t^{\xi}(x,y)\right\}$. This is the *Doob ξ-transform* technique commonly used in the framework of Dirichlet forms, Markov chains and axiomatic potential theory.

4.3 Subordinate Structures

Consider the Laplace operator Δ and the Schrödinger operator $\Delta_q = \Delta - q$ defined as $\Delta_q u(x) = \Delta u(x) - q(x)u(x)$, where $q(x) \geq 0$ in an infinite network X. Then, there are two families of superharmonic functions defined on X, one defined with respect to Δ and the other with respect to Δ_q. The knowledge of the relation between these two classes of superharmonic functions is useful in solving some classification problems in X. For example, suppose $u \geq 0$ is Δ-superharmonic on X. Then,

$$u(x) \geq \sum_{y \sim x} \frac{t(x,y)}{t(x)} u(y) \geq \sum_{y \sim x} \frac{t(x,y)}{t(x) + q(x)} u(y),$$

so that u is (Δ_q-superharmonic or) q-superharmonic on X. In this section, we compare two such families of superharmonic functions on X in a generalised way.

In a network X, we have defined that a function u is superharmonic on X if and only if $t(x)u(x) \geq \sum_{y \sim x} t(x,y)u(y)$ at every vertex x in X. Noting that $t(x) > 0$ at every vertex x, let us write $p(x,y) = \frac{t(x,y)}{t(x)}$ for any pair of vertices x and y in X. Then, u is superharmonic on x if and only if $u(x) \geq \sum_{y \sim x} p(x,y)u(y)$ at every vertex x in X. Let us say that $\{p(x,y)\}$ defines a *P-structure* on X. Note $0 \leq p(x,y) \leq 1$, $p(x,y) > 0$ if and only if $x \sim y$, and $p(x) = \sum_{y \neq x} p(x,y) = \sum_{y \sim x} p(x,y) = 1$ for any x in X.

Another system P' of transition indices $\{p'(x,y)\}$ is said to define a *structure P' subordinate* to P [8, p.297] if:

(a) $0 \leq p'(x,y) \leq p(x,y)$ for any pair x and y in X.
(b) $p'(x,y) > 0$ if and only if $x \sim y$.
(c) $p'(x,y) < p(x,y)$ for at least one pair x and y.

As an example, we consider an infinite tree with transition probabilities $\{p(x,y)\}$. Then the harmonic structure defined by the Schrödinger operator $\Delta_q u(x) = \Delta u(x) - q(x)u(x)$, where $q(x) \geq 0$ but $q(z) > 0$ for at least one vertex $z \in X$, is subordinate to the harmonic structure defined by the Laplace operator Δ.

Let us denote by (X, P) the network with the basic P-structure. The operator Δ is as usual defined by $\Delta u(x) = \sum_{y \sim x} p(x,y)[u(y) - u(x)] = -u(x) +$

$\sum_{y\sim x} p(x,y)u(y)$. If P' is a structure subordinate to P, then write $\Delta' u(x) = -u(x) + \sum_{y\sim x} p'(x,y)u(y)$. A function $u(x)$ defined on X is said to be P'-superharmonic (respectively, P'-harmonic) at a vertex z if and only if $\Delta' u(z) \leq 0$ (respectively, $\Delta' u(z) = 0$). A function u is said to be P'-superharmonic on a subset E if and only if u is defined on E and $\Delta' u(z) \leq 0$ at every vertex $z \in \overset{0}{E}$. A corresponding definition is given for P'-harmonic functions on E. With these definitions, it is immediate that the constant function 1 is a P'-superharmonic, but not P'-harmonic, function on X. Hence, there exists a P'-potential $v > 0$ on X, that is v is a positive P'-superharmonic on X whose greatest P'-harmonic minorant on X is 0 (see Property 5 below). Consequently, if a network (X, P) has a subordinate P'-structure, then the potential theory associated with the P'-structure resembles that of a hyperbolic network. For example, we can prove the following results as shown earlier:

1. For any vertex e in X, there exists a unique P'-potential $G'_e(x)$ on X such that $(-\Delta')G'_e(x) = \delta_e(x)$ for $x \in X$, (Theorem 3.2.6).
2. $G'_e(x) \leq G'_e(e)$ for all x in X, (Corollary 3.3.7).
3. If v is a P'-potential on X, then for x in X, $v(x) = \sum_{y\in X} (-\Delta')v(y)G'_y(x)$, (Theorem 3.3.1).
4. If $s(x) \geq 0$ is a P'-superharmonic function and $p(x)$ is a P'-potential with P'-harmonic support E and if $s(x) \geq p(x)$ for $x \in E$, then $s(x) \geq p(x)$ for all $x \in X$, (Theorem 3.3.6). In particular, if $p(x)$ is a P'-potential with finite P'-harmonic support A in X, then $p(x) \leq \max_{a\in A} p(a)$ for all x in X.
5. If s is P'-superharmonic on a set E and t is P'-subharmonic on E such that $s \geq t$ on E, then there exists the greatest P'-harmonic minorant (g.P'-h.m.) h of s, such that $s \geq h \geq t$ on E, (Theorem 2.4.10).
6. There always exists a (non-constant) positive P'-harmonic function on X, (Theorem 4.1.9).

Theorem 4.3.1. *Let E be a subset of the network (X, P) and let P' be a structure subordinate to P. Then, every P-potential on E is a P'-potential on E.*

Proof. Let $u > 0$ be a P-potential on X. Then, for every $y \in \overset{0}{E}$,

$$u(y) \geq \sum_{z\sim y} p(y,z)u(z) \geq \sum_{z\sim y} p'(y,z)u(z).$$

Hence, $u(x)$ is P'-superharmonic at $x = y$. Since y is arbitrary on $\overset{0}{E}$, u is P'-superharmonic on E. Let $h \geq 0$ be the g.P'-h.m. of u on E. Then, for $y \in \overset{0}{E}$,

$$h(y) = \sum_{z\sim y} p'(y,z)h(z) \leq \sum_{z\sim y} p(y,z)h(z).$$

Consequently, $h \geq 0$ is P-subharmonic on E. Since, u is a P-potential and the P-subharmonic $h \leq u$, we conclude that $h \equiv 0$. Hence, u is a $P^{'}$-potential on E. □

Theorem 4.3.2. *Let (X,P) be a network with positive P-potentials. Let $P^{'}$ be a structure subordinate to P. Then, for any vertex e, $G_e^{'}(x) \leq G_e(x)$ for every x in X.*

Proof. Since any positive P-superharmonic function on X is a $P^{'}$-superharmonic function on X, $G_e(x)$ is a $P^{'}$-superharmonic function on X. Now, for any $x \in X$,

$$\begin{aligned}(-\Delta^{'})G_e(x) &= G_e(x) - \sum_{z\sim x} p^{'}(x,z)G_e(z) \\ &\geq G_e(x) - \sum_{z\sim x} p(x,z)G_e(z) \\ &= (-\Delta)G_e(x) = \delta_e(x) = (-\Delta^{'})G_e^{'}(x).\end{aligned}$$

Hence, $s(x) = [G_e(x) - G_e^{'}(x)]$ is a $P^{'}$-superharmonic function. Since $-s(x) \leq G_e^{'}(x)$ on X, it follows that $-s \leq 0$ on X. Hence, $G_e(x) \geq G_e^{'}(x)$ for any x in X. □

Lemma 4.3.3. *Write $p^{'}(y) = \sum_{z\sim y} p^{'}(y,z)$ for each y in X. Then, $\sum_{y\in X} [1-p^{'}(y)]G_y^{'}(x) \leq 1$ for any $x \in X$.*

Proof. $s \equiv 1$ is a $P^{'}$-superharmonic function on X and

$$(-\Delta^{'})s(x) = s(x) - \sum_{y\sim x} p^{'}(x,y)s(y) = 1 - p^{'}(x).$$

Now, writing s as the unique sum of a positive $P^{'}$-potential and a non-negative $P^{'}$-harmonic function h, we have for any x in X,

$$\begin{aligned}1 = s(x) &= \sum_{y\in X} (-\Delta^{'})s(y)G_y^{'}(x) + h(x) \\ &= \sum_{y\in X} [1-p^{'}(y)]G_y^{'}(x) + h(x).\end{aligned}$$

Since h is non-negative, $\sum_{y\in X} [1-p^{'}(y)]G_y^{'}(x) \leq 1$ for any $x \in X$. □

Theorem 4.3.4. *For any network (X,P) with a subordinate structure $P^{'}$, the following are equivalent.*

1. $\sum_{y\in X} [1-p^{'}(y)]G_y^{'}(x) = 1$ *for some* $x \in X$.
2. $\sum_{y\in X} [1-p^{'}(y)]G_y^{'}(x) = 1$ *for every* $x \in X$.

3. *The constant function 1 is a $P^{'}$-potential on X.*
4. *The only bounded $P^{'}$-harmonic function on X is 0.*

Proof. By using the above lemma, write $1 = u(x) + h(x)$ where

$$u(x) = \sum_{y \in X} [1 - p^{'}(y)] G_y^{'}(x)$$

is a $P^{'}$-potential on X and $h(x)$ is a non-negative $P^{'}$-harmonic function on X.
1. $\Rightarrow$ 2. If $h(x) = 0$ at a vertex x, then $h \equiv 0$.
2. $\Rightarrow$ 3. Since $h \equiv 0$, $1 = u(x)$ is a $P^{'}$-potential on X.
3. $\Rightarrow$ 4. Let $b(x)$ be a bounded $P^{'}$-harmonic function on X. Let $|b(x)| \leq M$. Since M is a $P^{'}$-potential and $|b(x)|$ is a $P^{'}$-subharmonic function, $|b(x)| \leq 0$. That is, $b \equiv 0$.
4. $\Rightarrow$ 1. Write $1 = u(x) + h(x)$ as above. Since $0 \leq h(x) \leq 1$, the $P^{'}$-harmonic function $h \equiv 0$ by hypothesis. Hence, $1 = \sum_{y \in X} [1 - p^{'}(y)] G_y^{'}(x)$. □

Corollary 4.3.5. *Suppose there is no positive P-potential on X. Let $P^{'}$ be a structure subordinate to P. Then, the constant function 1 is a $P^{'}$-potential on X.*

Proof. Suppose h is a bounded $P^{'}$-harmonic function on X, $|h| \leq M$. Then, for $x \in X$,

$$\begin{aligned} |h(x)| &= \left| \sum_{y \sim x} p^{'}(x,y) h(y) \right| \\ &\leq \sum_{y \sim x} p^{'}(x,y) \, |h(y)| \\ &\leq \sum_{y \sim x} p(x,y) \, |h(y)| \, , \end{aligned}$$

so that $|h|$ is a bounded P-subharmonic function on X. Since, by assumption, there is no positive P-potential on X, $|h| = \alpha$, a constant. Since α is $P^{'}$-superharmonic and $|h|$ is $P^{'}$-subharmonic on X, then $|h| = \alpha$ should be P'-harmonic. But 1 is not $P^{'}$-harmonic, so that $\alpha = 0$ and hence $h \equiv 0$. Then, by the above theorem, the constant function 1 is a $P^{'}$-potential on X. □

Theorem 4.3.6. *Let (X, P) be a network and let $P^{'}$ be a structure subordinate to P. Then, the constant 1 is a $P^{'}$-potential if and only if any bounded $P^{'}$-superharmonic function u outside a finite set in X is of the form $u = p_1 - p_2$ outside a finite set where p_1 and p_2 are bounded $P^{'}$-potentials on X.*

Proof. Suppose 1 is a $P^{'}$-potential on X. Following the method used in the proof of Corollary 3.2.10, it can be seen that $u = q_1 - q_2 + v$ outside a finite set in X where q_1, q_2 are $P^{'}$-potentials with finite $P^{'}$-harmonic support and v is a $P^{'}$-superharmonic function on X. Since u, q_1, q_2 are all bounded outside a finite set, v is a bounded $P^{'}$-superharmonic function on X. Let $|v| \leq M$ on X. Then, $0 \leq v + M \leq 2M$ on X. Since the $P^{'}$-superharmonic function $v + M$ is majorized by the P'-potential $2M$, $v + M$ is a P'-potential on X. Write $u = (q_1 + v + M) - (q_2 + M) = p_1 - p_2$ outside a finite set where $p_1 = q_1 + (v + M)$ and $p_2 = q_2 + M$ are bounded P'-potentials on X.

On the other hand, assume that 1 is not a P'-potential on X. Suppose the constant function 1 which is P'-superharmonic function has a representation $1 = p_1 - p_2$ outside a finite set, where p_1, p_2 are bounded P'-potentials on X. Since $1 \leq p_1$ outside a finite set A and p_1 is bounded on A, there is a constant α such that $0 < \alpha < p_1$ on X. Since p_1 is a P'-potential on X, α and hence 1 should be a P'-potential on X, contradicting the assumption. □

Theorem 4.3.7. *Let (X, P) be an infinite network and P' be a structure subordinate to P. Then the constant function 1 is a P' -potential on X if and only if the following Maximum Principle is valid in X: Let F be any arbitrary proper subset of X. Let u be an upper bounded P'-subharmonic function on F such that $u \leq 0$ on ∂F. Then, $u \leq 0$ on F.*

Proof. Suppose 1 is a P'-potential on X. Let $u \leq 0$ on ∂F. Let $v = \sup(u, 0)$ on F extended by 0 outside F. Then, v is a P'-subharmonic function on X. Since u is upper bounded, for some $M > 0$, $v \leq M$ on X. Hence, $v \leq 0$ on X so that $v \equiv 0$ which implies that $u \leq 0$ on F.

On the other hand, suppose 1 is not a P'-potential on X. Then, there exists a bounded P'-harmonic function h on X. Take a vertex e and let $A = V(e)$ denote the set consisting of e and all its neighbors. Let $F = X \backslash \{e\}$. Then, $\overset{0}{F} = X \backslash A$ and $\partial F = A \backslash \{e\}$. Let $u = h - R_h^A$ which is a bounded P'-harmonic function on F, $u = 0$ on ∂F. Then, u should be the 0 function, if the Maximum Principle is valid; that is not the case. Thus, if the Maximum Principle is valid on X, then 1 has to be a P'-potential on X. □

We have defined the $\xi-$harmonic measure of the point at infinity in Sect. 4.2. When $\xi \equiv 1$, we shall simply refer to it as the harmonic measure of the point at infinity.

Theorem 4.3.8. *Let (X, P) be an infinite network and P' be a structure subordinate to P. Then, there exists a bounded positive P'-harmonic function on X if and only if the P'-harmonic measure of the point at infinity is positive.*

Proof. As in Sect. 4.2, let v be the P'-Dirichlet solution with boundary values 0 at e and 1 at the point at infinity. Recall that v is P'-subharmonic on X. If the harmonic measure of the point at infinity is positive, then for some vertex z in X, $v(z) > 0$. Then, as in Corollary 3.2.7, write $v = p_1 - p_2 + H$ outside a finite set in X, where H is P'-harmonic on X and p_1, p_2 are P'-potentials on X with finite P'-harmonic support and hence bounded on X. That is, there exists a P'-harmonic function H in X such that $|v - H| \leq M$ outside a finite set A in X. Since $v - H$ is bounded on A also, we shall assume without loss of generality that $|v - H| \leq M$ on X. Then, v is majorized by the P'-superharmonic function $H + M$ on X. Let h be the least P'-harmonic majorant of v on X. Since $0 \leq v \leq h \leq H + M$ on X, we have

$0 \leq h - v \leq (H + M) - v \leq 2M$. Since v is bounded on X, h is bounded on X; since $h(z) \geq v(z) > 0$, by the Minimum Principle, $h > 0$ on X. Thus, h is a bounded positive $P^{'}$-harmonic function on X.

Conversely, suppose there exists a bounded positive $P^{'}$-harmonic function on X. Then, by Theorem 4.3.4, the constant function 1 is not a $P^{'}$-potential. Hence there exists a $P^{'}$-harmonic function H on X such that $0 < H < 1$ on X. Taking the reduced function with respect to the subordinate structure $P^{'}$, if $u(x) = H(x) - R_H^e(x)$, then $0 < u < 1$ on $X \setminus \{e\}$, and $u(e) = 0$. Hence, if u_n is the $P^{'}$-Dirichlet solution on E_n with the values $u_n(e) = 0$ and $u_n(z) = 1$ if $z \in \partial E_n$ (see Sect. 4.2), then $u \leq u_n$. Consequently, by taking limits, $u \leq h$ where h is the $P^{'}$-harmonic measure of the point at infinity. Hence, the $P^{'}$-harmonic measure of the point at infinity is positive. □

Remark 4.3.1. In a network (X, P), let $P^{'}$ be a structure subordinate to P. Then $(X, P^{'})$ is always a hyperbolic network. All such networks $(X, P^{'})$ with subordinate structures fall into two categories depending on whether the constant 1 is a $P^{'}$-potential on X or not. Equivalently, this classification can be carried out depending on whether there exists or not a bounded positive $P^{'}$-harmonic function on X.

Chapter 5
Polyharmonic Functions on Trees

Abstract Biharmonic functions in the Euclidean space appear in the study of bending of plates or beams, a C^4- function u in a domain ω is biharmonic if $\Delta^2 u = 0$ in ω. Such functions can be considered in a network, but the operator Δ^2 is unwieldy. Another way to define biharmonic functions is used here: start with a harmonic function h and call u a biharmonic function generated by h if $\Delta u = h$ on X. This requires solving the Poisson equation $\Delta g = f$ when f is known. Unable to solve this equation in a general infinite network X, the network is restricted to a tree T in which every non-terminal vertex has at least two non-terminal neighbours. Then, it is possible to define inductively an m-harmonic function $(m \geq 2)$ u on T as a solution of the equation $\Delta u = v$ where v is an $(m-1)$-harmonic function. This chapter is about the potential theory associated with m-harmonic functions: m-superharmonic functions, m-potentials, domination principle for m-potentials, existence of m-harmonic Green functions, Riquier problem and the Riesz-Martin representation for positive m-superharmonic functions.

In the classical potential theory associated with the bending of plates or beams, we have an occasion to study biharmonic functions defined on an open set ω in $\Re^2$ or $\Re^3$. A C^4-function u defined on ω is said to be biharmonic on ω if $\Delta^2 u = \Delta(\Delta u) = 0$ on ω. More generally, a C^{2m}-function u on a domain ω in $\Re^n, n \geq 2$, is said to be a *polyharmonic function of order* m if $\Delta^m u = 0$ on ω. We shall refer to it as an *m-harmonic function* on ω. Many of the important properties of polyharmonic functions of finite order in $\Re^n$ are given in Nicolesco [56]. If u is an m-harmonic function in $\Re^n$(or more generally on a *star domain* D in $\Re^n$ with centre 0, that is, if $x \in D$ and $0 \leq \alpha \leq 1$, then $\alpha x \in D$), then there are m harmonic functions $h_1, h_2, ..., h_m$ in $\Re^n$ such that

$$u(x) = |x|^{2m-2} h_m(x) + |x|^{2m-4} h_{m-1}(x) + ... + |x|^2 h_2(x) + h_1(x).$$

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This basic representation, known as the *Almansi representation*, of u in $\Re^n$ is very useful in connecting the m-harmonic functions with harmonic functions. Thus, in Nicolesco [57] we find Liouville-Picard-Hadamard theorem for m-harmonic functions, Harnack-Montel theorems for families of m-harmonic functions, Generalised Green's formula, Riquier problem (which seeks to find an m-harmonic function u on a bounded domain ω, when the functions $u, \Delta u, ..., \Delta^{m-1}u$ are known on the boundary $\partial\omega$), m-harmonic Green functions of the first kind and the second kind etc. Aronszjan et al. [11] study polyharmonic functions of infinite order from the point of view of analytic functions.

Locally in $\Re^n$(that is, in a sphere for example), the properties of harmonic or biharmonic functions look the same whatever be the dimension $n \geq 2$. However, the global properties of harmonic functions alter much between $n = 2$ and $n \geq 3$; similarly for biharmonic functions between $2 \leq n \leq 4$ and $n \geq 5$. Based on this distinction, Sario et al. [60] develop a biharmonic classification theory of Riemannian manifolds analogous to the harmonic classification of Riemann surfaces. A primary concern is to check whether there is a biharmonic Green function defined on a given Riemannian manifold M. In this context, they introduce on a regular subregion of M, two biharmonic Green functions β and γ with a biharmonic fundamental singularity and with boundary conditions $\beta = \frac{\partial\beta}{\partial n} = 0$ and $\gamma = \Delta\gamma = 0$. Yamasaki [71] studies a discrete analogue to the biharmonic Green function γ and Kayano and Yamasaki [52] discuss an analogue to the biharmonic Green function β in an infinite network with symmetric conductance. Analogues of these biharmonic Green functions are also considered on homogeneous trees in Cohen et al. [38]. Another aspect of discrete polyharmonic functions appears in Cohen et al. [36] where on a homogeneous tree (and hence a tree with positive potentials) an integral representation (inspired by the Almansi representation in the classical case) is given for u, $\Delta^m u = 0$; and a one-one correspondence between polyharmonic functions and polymartingales is indicated.

In this context, we mention also an axiomatic study of biharmonic functions on a locally compact space Ω carried out by Smyrnélis [61] and [62]. He starts with a family H of coupled continuous functions (h_1, h_2) defined on open sets in Ω where h_1 and h_2 are related in a manner $(u, -\Delta u)$ is related to a biharmonic function u in the Euclidean space $\Re^n, n \geq 2$. Then imposing certain conditions for the local solvability of the Riquier problem (similar to the conditions for the local solvability of the Dirichlet problem in the Brelot-Bauer axiomatic treatment of harmonic functions on a locally compact space), Smyrnélis studies the properties of H-harmonic and H-potential couples on the lines of the axiomatic study of harmonic functions and potentials carried out by Bauer [17]. We mention also another interesting paper [25] which deals with a global study of polyharmonic functions in an axiomatic study.

In this chapter, we study the discrete version of m-harmonic functions on trees. In the classical case, we say that u is m-harmonic if $\Delta^m u = 0$ or inductively defining, u is m-harmonic if there exists an $(m-1)$-harmonic function v such that $\Delta u = v$. However, the operator Δ^m in the discrete case is unwieldy. So we reverse the order of appearance of u and v in the above definition of m-harmonic functions. For example, we say that a harmonic function h on a tree *generates* a biharmonic function b if $\Delta b = h$. This development requires a harmonic extension theorem which we are able to prove only on trees T and not on networks that may contain cycles. Once this harmonic extension theorem is established, it is easy to prove that the Poisson equation $\Delta u = f$ can be solved for u, given an arbitrary real-valued function f. Consequently, any real-valued function on T is the difference of two suprharmonic functions on T. Then we can introduce, on functions on trees, the notions like m-superharmonic functions, m-potentials, balayage for positive m-superharmonic functions, m-harmonic Green functions etc. and study their properties.

5.1 Polyharmonic Functions on Infinite Trees

Let f be a real-valued function on a network. To find a solution u for the equation $\Delta u = f$ (known as the Poisson equation) is a basic requirement for the way the theory of polyharmonic functions is developed in this chapter. However, we know that (Theorem 2.2.6) in a finite network X with symmetric conductance, $\Delta u = f$ has a solution if and only if $\sum_{x \in X} f(x) = 0$. This indicates that certain conditions on the network and on the generating function f are needed to solve the equation $\Delta u = f$. Here we restrict our study of polyharmonic functions to a specific class of infinite trees, possibly with non-symmetric conductance.

In a network X, recall that a closed path $\{x_1, x_2, ..., x_n\}$ where $x_1, x_2, ..., x_{n-1}$ are distinct vertices and $x_1 = x_n, n > 3$, is called a cycle. A tree T is an infinite network, connected, locally finite without any self-loop or cycles. A vertex e in T is terminal if it has only one neighbour in T. Since T is infinite, a non-terminal vertex in T has to have at least one non-terminal vertex as its neighbour. *We assume in this chapter that every non-terminal vertex has at least two non-terminal neighbours.* Star-shaped trees with a central vertex, homogeneous trees and more generally trees without terminal vertices are examples of infinite trees which satisfy this assumption. Remark that if $t(x, y)$ is the conductance in T, then we can set $p(x, y) = \frac{t(x,y)}{t(x)}$ for every pair of vertices x and y. Consequently, let us denote the conductance on T by $p(x, y) \geq 0$ for which $p(x, y) > 0$ if and only if $x \sim y$, and $\sum_{y \sim x} p(x, y) = 1$ for every vertex x. It is possible that $p(x, y) \neq p(y, x)$. For any real-valued function u on T, we write

$$\Delta u(x) = \sum_{y \sim x} p(x, y)[u(y) - u(x)] = \left[\sum_{y \sim x} p(x, y)u(y)\right] - u(x).$$

As earlier, if u is a real-valued function defined on a subset E of T, then u is said to be superharmonic (respectively, harmonic or subharmonic) on E if $\Delta u(x) \leq 0$ (respectively, $\Delta u(x) = 0$ or $\Delta u(x) \geq 0$) for every $x \in \overset{0}{E}$.

Lemma 5.1.1. *Let E be a connected set in T and $F = V(E)$. Then, any $z \in \partial F$ has only one neighbour in F.*

Proof. Let $z \in \partial F$. Then, z has a neighbour $x_0 \in E$. Suppose $y \in F$, $y \neq x_0$, and $y \sim z$. If $y \in E$, then there exists a path $\{x_0, x_1, ..., x_n = y\}$ in E, since E is connected. Then, $\{z, x_0, x_1, ..., y, z\}$ is a cycle in T, not possible. On the other hand, if $y \in F \backslash E$, then y has a neighbour a in E. If $a = x_0$, then $\{y, a = x_0, z, y\}$ is a cycle, not possible; if $a \neq x_0$, then there is a path $\{a, y_1, y_2, ..., x_0\}$ connecting a and x_0 in E, so that $\{y, a, y_1, ..., x_0, z, y\}$ is a cycle, not possible. Thus, there is no $y \in F$ such that $y \neq x_0$ and $y \sim z$. That is, x_0 is the only neighbour of z in F. □

Theorem 5.1.2. *Let E be an arbitrary connected subset of T and $F = V(E)$. Suppose u is a superharmonic function defined on F. Then, there exists a superharmonic function v on T such that $v = u$ on F and $\Delta v(x) = 0$ for each $x \in T \backslash \overset{0}{F}$.*

Proof. Let $z \in \partial F$. By the above Lemma 5.1.1, z has only one neighbour x_0 in F. Note that z is not a terminal vertex; for, z has a neighbour in E, which by the above-mentioned uniqueness should be x_0. If z were a terminal vertex, then $x_0 \in E \subset \overset{0}{F}$ being the only neighbour of z, by definition z should be in $\overset{0}{F}$, a contradiction. Since, by the assumption on T, the non-terminal vertex z should have at least two non-terminal vertices as neighbours, z should have at least one non-terminal vertex as a neighbour outside F. Let $A = \{y_1, y_2, ..., y_k\}$ be the neighbours of z outside F. Let A_1 denote the set of terminal vertices in A and let $A_2 = A \backslash A_1$. Then, $A_2 \neq \phi$.

We shall extend the given function u from F to a set that includes A. Define $v(x) = u(x)$ if $x \in F$; if $y \in A_1$, then take $v(y) = u(z)$; if $y \in A_2$, then take $v(y) = \lambda$, a constant. Choose the constant λ such that $\Delta v(z) = 0$; that is,

$$v(z) = u(z) = p(z, x_0)u(x_0) + u(z) \sum_{y \in A_1} p(z, y) + \lambda \sum_{y \in A_2} p(z, y).$$

This procedure can be used with respect to each one of the vertices on ∂F to get an extension of u from F to a function v on the set $V(F)$ such that $\Delta v = 0$ at each one of the vertices on ∂F. Thus v is a function defined on $V(F) \supset F$ such that $v = u$ on F and $\Delta v(x) = 0$ for each $x \in \partial F$. Then, v is similarly extended to $V[V(F)]$. Since T is connected, eventually v is defined at any vertex x in T such that $\Delta v(x) = 0$ at each $x \in T \backslash \overset{0}{F}$ and $v = u$ on F. □

Remark 5.1.1. In the proof of the above Theorem 5.1.2, the assumption that u is superharmonic on F is useful only to conclude that v is superharmonic on T. Consequently, a slightly more general form of the statement of the theorem is: *Let E be an arbitrary connected subset of T and $F = V(E)$. Suppose u is a real-valued* *function defined on F. Then there exists a function v on T such that $v = u$ on F and $\Delta v(x) = 0$ for each $x \in T \setminus \overset{0}{F}$.*

Corollary 5.1.3. *Let E be an arbitrary connected subset of T and $F = V(E)$. Suppose u is a harmonic function defined on F. Then, there exists a harmonic function v on T such that $v = u$ on F.*

Remark 5.1.2. 1. Measuring distances from a fixed vertex e and taking $F = \{x : |x| \leq n\}$, Bajunaid et al. [15, Proposition 4.2] prove the above harmonic extension in a tree T, *ignoring the Laplacian at the terminal vertices in* T. Note that if $E = \overset{0}{F}$, then E is connected and $F = V(E)$.

2. Using the above theorem, it is easy to show that there are superharmonic functions with point harmonic support on T (Theorems 3.2.6 and 3.4.5): Let e be a fixed vertex and $F = V(e)$. Let u be the function on F such that $u(e) = 1$ and $u(x) = 0$ if $x \in F \setminus \{e\}$. Then, proceeding as in the proof of the above theorem, we construct a function v on T such that $v = u$ on F and $\Delta v(x) = 0$ for each $x \in T \setminus \{e\}$. Consequently,

$$\Delta v(e) = \sum_{x \sim e} p(e, x)[v(x) - v(e)] = -v(e) = -1,$$

that is v is superharmonic on T with harmonic support at e and $\Delta v(x) = -\delta_e(x)$ for each x in T.

3. If the assumption that each non-terminal vertex has at least two non-terminal vertices is not verified, the above theorem may not be valid. Consider, for example, the following situation: Let $T = \{a, b, x_1, x_2, ...\}$ be an arrow-shaped infinite tree such that $x_i \sim x_{i+1}$ for $i \geq 1$, a and b are terminal vertices, $a \sim x_1$ and $b \sim x_1$. Let $p(x_1, a) = \frac{1}{4} = p(x_1, b)$ and $p(x_i, x_{i+1}) = p(x_{i+1}, x_i) = \frac{1}{2}$ for $i \geq 1$. Let $E = \{x_2, x_3\}$ which is connected. Then, $F = V(E) = \{x_1, x_2, x_3, x_4\}$ which is connected and circled and $\overset{0}{F} = E$. Let $u(x_k) = k$ for $1 \leq k \leq 4$. Then, u is harmonic on F.

   Suppose there exists a harmonic function v on T such that $v = u$ on F. Then, $v(a) = 1 = v(b)$ since $\Delta v(a) = 0 = \Delta v(b)$ and a, b are terminal vertices. But then, v cannot be harmonic at x_1 since

$$\Delta v(x_1) = \frac{1}{4}[v(a) - v(x_1)] + \frac{1}{4}[v(b) - v(x_1)] + \frac{1}{2}[v(x_2) - v(x_1)] = \frac{1}{2}.$$

   That is, there is no harmonic extension for u to the whole of T. Note that the non-terminal vertex x_1 has only one non-terminal vertex, namely x_2, as its neighbour.

4. Again, the above theorem may not be valid in a general network with cycles. For example, consider the network X consisting of two infinite branches $\{a, x_1, x_2, ...\}$ and $\{a, y_1, y_2, ...\}$ with a as the only common vertex, each vertex except x_2 and y_2 having only two neighbours, and $x_2 \sim y_2$ so that each of the vertices x_2 and y_2 has 3 neighbours. Suppose the conductance on each edge is $\frac{1}{2}$.

Take $E = X \setminus \{a, x_1, y_1\}$ along with all the edges connecting the vertices in E. Then E is a connected subset of X, $\partial E = \{x_2, y_2\}$, and $F = V(E) = X \setminus \{a\}$. Let u be the function defined on F such that $u(x_k) = 3 + 2k$, $u(y_k) = 7 + k$, for $k \geq 2$, $u(x_1) = 3$ and $u(y_1) = 10$. Then, $\Delta u(x) = 0$ for each $x \in E = \overset{0}{F}$. Hence, u is harmonic on F. However, it is not possible to find a harmonic function v on X such that $v = u$ on F. For, suppose such a function v exists on X. Then,

$$0 = \Delta v(x_1) = \frac{1}{2}\left[v(x_2) - v(x_1)\right] + \frac{1}{2}\left[v(a) - v(x_1)\right]$$

$$= \frac{1}{2}\left[7 - 3\right] + \frac{1}{2}\left[v(a) - 3\right], \text{ so that } v(a) = -1. \text{ Again,}$$

$$0 = \Delta v(y_1) = \frac{1}{2}\left[v(y_2) - v(y_1)\right] + \frac{1}{2}\left[v(a) - v(y_1)\right]$$

$$= \frac{1}{2}\left[9 - 10\right] + \frac{1}{2}\left[-1 - 10\right] = -6, \text{not valid.}$$

Hence, the harmonic extension of u from F to a harmonic function v in X is not possible.

Theorem 5.1.4. *Let f be a non-negative function on T. Then, there exists a superharmonic function s on T such that $\Delta s(x) = -f(x)$, for each x in T.*

Proof. For any given vertex e in T, let $q_e(x)$ denote a superharmonic function on T such that $\Delta q_e(x) = -\delta_e(x)$ for each X in T.

First, suppose that f has finite support, that is $f = 0$ outside a finite set A. Then, $s(x) = \sum\limits_{a \in A} f(a) q_a(x)$ is a superharmonic function on T such that $\Delta s(x) = -f(x)$ for $x \in T$. Suppose now that $f \geq 0$ is arbitrary. Let E_1 be a finite connected subset of T. Let $E_{n+1} = V(E_n)$ for $n \geq 1$. Then each E_n is a finite connected circled subset of T, such that $E_n \subset \overset{\circ}{E}_{n+1}$. Since T is connected, we also find that $T = \bigcup\limits_{n \geq 1} E_n$. Let f_0 be the restriction of f on E_1 and f_n be the restriction of f on $E_{n+1} \setminus E_n$ for $n \geq 1$. Let $u_n(x), n \geq 0$, be a superharmonic function on T such that $\Delta u_n(x) = -f_n(x)$ for $x \in T$. When $n \geq 2$, since u_n is harmonic at each vertex in E_n, by the above Corollary 5.1.3, we can find a harmonic function v_n on T such that $v_n = u_n$ on E_n.

Let $s(x) = u_0(x) + u_1(x) + \sum\limits_{n \geq 2} [u_n(x) - v_n(x)]$. This is a superharmonic function on T. For, if K is any finite set in T, then there exists some m such that $K \subset E_n$ for $n \geq m$. Consequently, for any vertex $a \in K$, the infinite sum corresponding to $s(a)$ contains only a finite number of non-zero terms. Thus, $s(x)$ is finite at every vertex $x \in T$, so that s defines a superharmonic function on T. Further, $\Delta s(x) = \Delta u_0(x) + \Delta u_1(x) + \sum_{n \geq 2} \Delta[u_n(x) - v_n(x)] = -f(x)$. □

Remark 5.1.3. 1. To prove the existence of a solution s for the Poisson equation $\Delta s = -f$ in the above theorem, we have used the harmonic extension Theorem 5.1.2 which requires the assumption that in T every non-terminal vertex has at least two non-terminal vertices as neighbours. However, this assumption is not a necessary condition for the existence of a solution s. For example, consider the arrow-shaped infinite tree $T = \{a, b, x_0, x_1, ...\}$ with a and b as terminal vertices, $a \sim x_0, b \sim x_0$. If $n \geq 1$, then each x_n has two neighbours x_{n-1} and x_{n+1}. But among the three neighbours of x_0, only x_1 is non-terminal. Thus, T does not satisfy the stated assumption with respect to x_0.

Let $p(a, x_0) = 1 = p(b, x_0)$, $p(x_0, a) = \frac{1}{4} = p(x_0, b)$, $p(x_0, x_1) = \frac{1}{2}$ and $p(x_n, x_{n+1}) = p(x_n, x_{n-1}) = \frac{1}{2}$ for $n \geq 1$.

Let $f \geq 0$ be any real-valued function on T. For any constant c, set $u(x_0) = c$, $u(a) = c + f(a)$, $u(b) = c + f(b)$, $u(x_1) = c - \frac{1}{2}[f(a) + f(b)] - 2f(x_0)$ and $u(x_{n+1}) = 2u(x_n) - u(x_{n-1}) - 2f(x_n)$ for $n \geq 1$. Then, $\Delta u(x) = -f(x)$ for any $x \in T$.

2. Any real-valued function g on T is a δ-superharmonic function (that is, g is a difference of two superharmonic functions) on T. For, write $\Delta g = f$. By Theorem 5.1.4, there exist subharmonic functions u_1 and u_2 on T such that $\Delta u_1 = f^+$ and $\Delta u_2 = f^-$. Write $u = u_1 - u_2$ so that $\Delta u = f = \Delta g$. Since $\Delta(g - u) = 0$, we have $g = u + h$ where h is a harmonic function on T. Hence $g = -u_2 - [-(u_1 + h)]$ is a difference of two superharmonic functions on T.

Theorem 5.1.5. *Let F be any subset of T and $f \geq 0$ be defined on F. Then, there exists a superharmonic function s on T such that $\Delta s(x) = -f(x)$ for each x in F.*

Proof. Extend f as a non-negative function g on T, by taking $g = 0$ outside F. Then, by the above theorem, there exists a superharmonic function s on T such that $\Delta s = -g$ on T. In particular, $\Delta s(x) = -f(x)$ if $x \in F$. □

Consequently, for any real-valued function f on a subset F in T, there exists a δ-superharmonic function u on T such that $\Delta u(x) = -f(x)$ for each $x \in F$. By the same procedure, we can find a δ-superharmonic function v on T such that $\Delta v = -u$ on F, which can be written as $(-\Delta)^2 v = f$ on F. Continuing thus, for any $m \geq 1$, we can find a δ-superharmonic function s on T such that $(-\Delta)^m s(x) = f(x)$ for each $x \in F$.

Definition 5.1.1. Let $(s_i)_{m \geq i \geq 1}$ be a set of m real-valued functions on a set F in T such that $(-\Delta)s_i = s_{i-1}$ on $\overset{0}{F}$ for $m \geq i \geq 2$. Then, $s = (s_i)_{m \geq i \geq 1}$ is said to be an m-superharmonic (respectively, m-harmonic, m-subharmonic) function on F if and only if s_1 is superharmonic (respectively harmonic, subharmonic) on F. We say that the m-superharmonic (respectively m-harmonic, m-subharmonic) function $s = (s_i)_{m \geq i \geq 1}$ is generated by s_1.

Notation. 1. If $s = (s_i)_{m \geq i \geq 1}$ and $t = (t_i)_{m \geq i \geq 1}$, then we write $s + t = (s_i + t_i)_{m \geq i \geq 1}$, $\alpha s = (\alpha s_i)_{m \geq i \geq 1}$ and say that $s \geq t$ if and only if $s_i \geq t_i$ for each i, in particular $s \geq 0$ if and only if $s_i \geq 0$ for each i, $m \geq i \geq 1$.

2. For $k > m$, an m-harmonic function $h = (h_m, ..., h_1)$ can be identified as a k-harmonic function $h = (h_m, ..., h_1, 0, 0, ..., 0)$ by adding $(k - m)$ zeros.

Theorem 5.1.6. *Let E be a connected subset of T and $F = V(E)$. Let $h = (h_i)_{m \geq i \geq 1}$ be an m-harmonic function defined on F. Then, there exists an m-harmonic function $H = (H_i)_{m \geq i \geq 1}$ on T such that $H_i(x) = h_i(x)$ for $x \in F$ and $m \geq i \geq 1$.*

Proof. Note that h_1 is harmonic on F. Hence by Corollary 5.1.3, there exists a harmonic function H_1 on T such that $H_1 = h_1$ on F. Let $(-\Delta)u = H_1$ on T. Write $t(x) = u(x) - h_2(x)$, if $x \in F$. Then, for $y \in \overset{0}{F}$,

$$(-\Delta)t(y) = (-\Delta)u(y) - (-\Delta)h_2(y) = H_1(y) - h_1(y) = 0.$$

Hence, $t(x)$ is harmonic on F. Then, we can find a harmonic function v on T such that $v(x) = t(x)$ on F. Write $H_2(x) = u(x) - v(x)$, for each x in T. Then, $(-\Delta)H_2(x) = H_1(x)$ for each $x \in T$ and $H_2(x) = h_2(x)$ for each $x \in F$. Proceeding thus, we construct an m-harmonic function $H = (H_i)_{m \geq i \geq 1}$ on T such that $H_i(x) = h_i(x)$ for $x \in F$ and $m \geq i \geq 1$. □

In the complex plane, if $f(z)$ is analytic in $r < |z| < R$, then it has the Laurent series representation $f(z) = \sum_{-\infty}^{+\infty} a_n z^n$. Here $f_1(z) = \sum_{-\infty}^{-1} a_n z^n$ is analytic on $|z| > r$ with a removable singularity at the point at infinity and $f_2(z) = \sum_{0}^{+\infty} a_n z^n$ is analytic in $|z| < R$, such that $f(z) = f_1(z) + f_2(z)$ on $r < |z| < R$. This Laurent decomposition is a very useful representation while studying the singularities of analytic functions. Similar representations for harmonic functions in $\Re^n, n \geq 2$, are also well-known (for example, [13]).

Let k be a compact set contained in an open set ω in $\Re^n, n \geq 2$. Let h be a harmonic function on $\omega \backslash k$. Then, h can be written in the form $h = s + t$ on $\omega \backslash k$, where s is harmonic on $\Re^n \backslash k$ and t is harmonic on ω. Moreover, this representation is unique if we impose the following restrictions:

1. If $n \geq 3$, $\lim_{x \to \infty} s(x) = 0$.
2. If $n = 2$, $\lim_{x \to \infty} [s(x) - \alpha \log |x|] = 0$ for some constant α.

The following theorems give the Laurent decomposition for harmonic and polyharmonic functions on T.

Theorem 5.1.7. *(Laurent decomposition for harmonic functions) Let T be an infinite tree in which every non-terminal vertex has at least two non-terminal neighbours. Let E be a finite connected subset of T and $F = V(E)$. Let A be a non-empty subset in E. Suppose u is a harmonic function on $F \backslash \overset{0}{A}$. Then, there exist a harmonic function s on $T \backslash \overset{0}{A}$ and a harmonic function t on F such that $u = s + t$ on $F \backslash \overset{0}{A}$. Moreover,*

i. *If there are positive potentials on T, then s and t are uniquely determined if we impose the restriction $|s| \leq p$ outside a finite set, where p is a positive potential on T.*
ii. *If there are no positive potentials on T, then s and t are uniquely determined up to an additive constant, if we impose the restriction that for some α, $[s - \alpha H]$ is bounded outside a finite set where $H(x)$ is the analogue of $\log|x|$ in T (namely, $H \geq 0$ on T, unbounded, $\Delta H(x) = \delta_e(x)$ where e is a fixed vertex and $H > 0$ outside a finite set in T).*

Proof. Extend u arbitrarily on $\overset{0}{A}$ as a real-valued function. Then, by the Remark following Theorem 5.1.2, there exists a function v on $T \setminus \overset{0}{A}$ such that $v = u$ on $F \setminus \overset{0}{A}$ and $\Delta v(x) = 0$ at every vertex x in the interior of $T \setminus \overset{0}{A}$.

i. Suppose there are positive potentials on T. Then, there exist potentials p_1, p_2 on T and a harmonic function f on T such that $v = f + p_1 - p_2$ outside a finite set in T (Corollary 3.2.7). Write $s = v - f$ on $T \setminus \overset{0}{A}$ and $t = f$ on F. Then, $|s| \leq p_1 + p_2 = p$ outside a finite set, t is harmonic on F and $u = v = s + t$ on $F \setminus \overset{0}{A}$. As for the uniqueness of decomposition, suppose $u = s^{'} + t^{'}$ is another such representation with $\left|s^{'}\right| \leq p^{'}$ outside a finite set, where $p^{'}$ is a potential on T. Then, the function h defined on T such that $h = s - s^{'}$ on $T \setminus \overset{0}{A}$ and $h = t^{'} - t$ on F is a well-defined harmonic function such that $|h| \leq p + p^{'}$ outside a finite set. Hence $h \equiv 0$. This proves that $s = s^{'}$ and $t = t^{'}$.
ii. Suppose there are no positive potentials on T, that is T is a parabolic tree. Then, by Proposition 3.4.7, $v = f + \alpha H + b$ outside a finite set, where f is harmonic on T and b is bounded harmonic outside a finite set. Let $s = v - f$ on $T \setminus \overset{0}{A}$ and $t = f$ on F. Then, $[s - \alpha H]$ is bounded outside a finite set and $u = v = s + t$ on $F \setminus \overset{0}{A}$. As for the uniqueness, suppose $u = s^{'} + t^{'}$ on $F \setminus \overset{0}{A}$ is another such representation, with $[s^{'} - \alpha^{'} H]$ being bounded outside a finite set. Then, the function h defined on T such that $h = s - s^{'}$ on $T \setminus \overset{0}{A}$ and $h = t^{'} - t$ on F is a well-defined harmonic function on T such that $\left|h - (\alpha - \alpha^{'})H\right| \leq |s - \alpha H| + \left|s^{'} - \alpha^{'} H\right|$ is bounded outside a finite set. This is possible only if $\alpha - \alpha^{'} = 0$, since H is an unbounded non-negative function and since a harmonic function on a parabolic tree is constant if it is bounded on one side. Consequently, the harmonic function h on the parabolic tree is bounded and hence a constant. □

Theorem 5.1.8. *(Laurent decomposition for m-harmonic functions) Let T be an infinite tree as in Theorem 5.1.7. Let E be a finite connected set of T and $F = V(E)$. Let A be a non-empty subset of E. Suppose $u = (u_i)_{m \geq i \geq 1}$ is an m-harmonic function on $F \setminus \overset{0}{A}$. Then, there exist an m-harmonic function s on $T \setminus \overset{0}{A}$ and an m-harmonic function t on F such that $u = s + t$ on $F \setminus \overset{0}{A}$. This representation can* *be chosen to be unique up to an additive $(m-1)$-harmonic function on T if T has positive potentials; otherwise, the representation is unique up to an m-harmonic function generated by a constant.*

Proof. Since u_1 is harmonic on $F\setminus \overset{0}{A}$, there exist a harmonic function s_1 on $T \setminus \overset{\circ}{A}$ and a harmonic function t_1 on F such that $u_1 = s_1 + t_1$ on $F\setminus \overset{0}{A}$. Choose two functions f_1 and g_1 on T such that $(-\Delta)f_1 = s_1$ on $T\setminus \overset{0}{A}$ and $(-\Delta)g_1 = t_1$ on F. Then, $(-\Delta)u_2(x) = u_1(x) = (-\Delta)f_1(x) + (-\Delta)g_1(x)$ on $F\setminus \overset{0}{A}$, so that $u_2(x) = f_1(x)+g_1(x)+h_1(x)$, where h_1 is harmonic on $F\setminus \overset{0}{A}$. Then, by the above Theorem 5.1.7, there exist f_2 harmonic on $T\setminus \overset{0}{A}$ and g_2 harmonic on F such that $h_1 = f_2 + g_2$ on $F\setminus \overset{0}{A}$. Write $s_2 = f_1 + f_2$ and $t_2 = g_1 + g_2$. Then, $(-\Delta)s_2 = s_1$ on $T\setminus \overset{0}{A}$, $(-\Delta)t_2 = t_1$ on F, and $u_2 = s_2 + t_2$ on $F\setminus \overset{0}{A}$.

Proceeding in a similar manner, we construct $s = (s_i)_{m\geq i\geq 1}$ which is m-harmonic on $T\setminus \overset{0}{A}$ and $t = (t_i)_{m\geq i\geq 1}$ which is m-harmonic on F, such that $u = s+t$ on $F\setminus \overset{0}{A}$. As for the uniqueness, recall that as in the above Theorem 5.1.7, $s_1$and t_1 can be uniquely determined, if T has positive potentials; otherwise, s_1, t_1 are unique up to an additive constant. Consequently, if $u = s' + t'$ is another such representation, then the function $v = (v_i)_{m\geq i\geq 1}$ defined on T such that $v = s - s'$ on $T\setminus \overset{0}{A}$ and $v = t' - t$ on F is a well-defined function on T such that $v_1 \equiv 0$, if T has positive potentials and v_1 is a constant if T has no positive potentials. Hence, v is $(m-1)$-harmonic on T, if T has positive potentials; otherwise, v is an m-harmonic function on T generated by a constant. □

Theorem 5.1.9. *Let T be an infinite tree as in Theorem 5.1.7. Let s be an m-superharmonic function on a subset A in T and let t be an m-subharmonic function on A such that $t \leq s$ on A. Then, there exists an m-harmonic function h on A such that $t \leq h \leq s$ on A, with the additional property that if $h^\bullet$ is any m-harmonic function on A such that $h^\bullet \leq s$ on A, then $h^\bullet \leq h$ on A.*

Proof. Let $s = (s_i)_{m\geq i\geq 1}$ and $t = (t_i)_{m\geq i\geq 1}$. Let $\Im$ be the family of subharmonic functions u on A such that $t_1 \leq u \leq s_1$. Let $h_1(x) = \sup_{u\in\Im} u(x)$. Then, h_1 is harmonic on A and it is the greatest harmonic minorant of s_1 on A. Let f be the function defined on T such that $f = h_1$ on A extended by 0 outside A. Then, there exists a δ-superharmonic function g on T such that $(-\Delta)g = f$. Let H_2 be the restriction of g on the subset A. Then, $(-\Delta)H_2 = h_1$ on $\overset{0}{A}$. Similarly, choose f_2 and g_2 on A such that $(-\Delta)f_2 = s_1 - h_1$ on $\overset{0}{A}$ and $(-\Delta)g_2 = t_1 - h_1$ on $\overset{0}{A}$. Note that f_2 is superharmonic on A and g_2 is subharmonic on A. Moreover, $(-\Delta)s_2 = s_1 = (-\Delta)f_2+(-\Delta)H_2$ on $\overset{0}{A}$ and $(-\Delta)t_2 = t_1 = (-\Delta)g_2+(-\Delta)H_2$ on $\overset{0}{A}$. Consequently, $s_2 = f_2 + H_2+$ (a harmonic function) on A. Write $s_2 = f_2' + H_2$ where f_2' is superharmonic on A. Similarly, write $t_2 = g_2' + H_2$ where g_2'

is subharmonic on A. Since $s_2 \geq t_2$ by hypothesis, we have $f_2^{'} \geq g_2^{'}$. Let $h^{'}$ be the greatest harmonic minorant of $f_2^{'}$ on A. Let $h_2 = h^{'} + H_2$. Then, $(-\Delta)h_2 = h_1$ on $\overset{0}{A}$ and $s_2 \geq h_2 \geq t_2$ on A.

Suppose $h_2^{'}$ is another function on A such that $s_2 \geq h_2^{'}$ on A and $(-\Delta)h_2^{'} = h_1$ on $\overset{0}{A}$. Then, $h_2^{'} = h_2+$ (a harmonic function v on A) $= (h^{'} + H_2) + v$. Since $s_2 \geq h_2^{'}$ on A, we should have $f_2^{'} \geq h^{'} + v$ on A. Since $h^{'}$ is the g.h.m. of $f_2^{'}$ on A, we have $v \leq 0$ on A. Hence, $h_2^{'} \leq h_2$.

A similar procedure yields an m-harmonic function $h = (h_i)_{m \geq i \geq 1}$ on A such that $s \geq h \geq t$ on A. This function has the additional property that if $h^\bullet$ is any m-harmonic function on A such that $s \geq h^\bullet \geq t$ on A, then $h \geq h^\bullet$. □

Recall that in this chapter, T stands for an infinite tree in which each non-terminal vertex has at least two non-terminal neighbours.

Definition 5.1.2. Let s be an m-superharmonic function defined on a subset A in T. Suppose there exists an m-subharmonic function t on A such that $s \geq t$ on A. Then the m-harmonic function h constructed as in the above Theorem 5.1.9 such that $s \geq h \geq t$ on A is called the greatest m-harmonic minorant (g.m-h.m.) of s in A.

Theorem 5.1.10. *Let $0 < s = (s_i)_{m \geq i \geq 1}$ be an m-superharmonic function on T, $m \geq 2$. Let u_1 be a superharmonic function on T such that $0 < u_1 \leq s_1$. Then there exists an m-superharmonic function $u = (u_i)_{m \geq i \geq 1}$ on T such that for $2 \leq i \leq m$, each u_i is a potential and $u_i \leq s_i$ on T.*

Proof. When $m \geq 2$, the existence of an m-superharmonic function $s > 0$ on T implies that there are positive potentials on T. Let $(-\Delta)v_1 = u_1$ and $(-\Delta)v_2 = s_1 - u_1$. Then v_1 and v_2 are superharmonic and $(-\Delta)(v_1 + v_2) = s_1 = (-\Delta)s_2$, so that $s_2 = v_1 + v_2 + h_2$, where h_2 is a harmonic function on T. Since $s_2 > 0$, v_1 majorizes the subharmonic function $-(v_2+h_2)$ so that v_1 is the sum of a potential u_2 and a harmonic function. Consequently, $(-\Delta)u_2 = (-\Delta)v_1 = u_1$. Note that since $u_1 > 0$, v_1 cannot be harmonic and hence $u_2 > 0$ on T. Write $s_2 =$ (a potential p_2) + (a non-negative harmonic function), so that $p_2 = u_2+$ (a superharmonic function q) on T. Since u_2 is a potential and $u_2 \geq -q$, we should have $-q \leq 0$. Hence $u_2 \leq p_2 \leq s_2$. Repeated use of this procedure yields the announced result. □

Theorem 5.1.11. *(Minimum Principle for m-superharmonic functions) Let E be a finite set in T. Let $s = (s_i)_{m \geq i \geq 1}$ be an m-superharmonic function on E and $h = (h_i)_{m \geq i \geq 1}$ be an m-subharmonic function on E such that $s \geq h$ on ∂E. Then $s \geq h$ on E.*

Proof. Since s_1 is superharmonic on E and h_1 is subharmonic on E such that $s_1 \geq h_1$ on ∂E by hypothesis, $s_1 \geq h_1$ on E (Corollary 1.4.3). Write $v_2 = s_2 - h_2$ on E. Then, on $\overset{0}{E}$, $(-\Delta)v_2 = (-\Delta)s_2 - (-\Delta)h_2 = s_1 - h_1 \geq 0$. Hence, v_2 is superharmonic on E and by assumption $v_2 \geq 0$ on ∂E. Hence $v_2 \geq 0$ on E, that is $s_2 \geq h_2$ on E. By recurrence, $s_i \geq h_i$ on E for all i, $1 \leq i \leq m$; that is, $s \geq h$ on E. □

Corollary 5.1.12. *Let $h = (h_i)_{m \geq i \geq 1}$ be an m-harmonic function defined on a finite set E in T. If $h = 0$ on ∂E, then $h \equiv 0$.*

Riquier Problem. Let D be a bounded domain in the Euclidean space $\Re^n, n \geq 2$. Suppose $f_1, f_2, ..., f_m$ are m finite continuous functions on the boundary ∂D. Then the Riquier problem is to determine a function u on $\overline{D}$ such that u is a polyharmonic function of order m on D and $\lim_{x \to z, x \in D} (-\Delta)^{i-1} u(x) \to f_i(z)$, for every z on ∂D, $1 \leq i \leq m$, $(-\Delta)^0 = 1$. This boundary value problem has a unique solution [57, p.28] if D is a Dirichlet domain, that is the Dirichlet problem for finite continuous boundary values is solvable in D. We solve now a discrete analogue of this problem.

Theorem 5.1.13. *Let F be a finite subset of T, and $E \subset \overset{0}{F}$. For $1 \leq i \leq m$, let f_i be a bounded function on $F \backslash E$. Then there exist m uniquely determined functions h_i on F such that $h_i = f_i$ on $F \backslash E$ and $h = (h_i)_{m \geq i \geq 1}$ is an m-harmonic function at each vertex of E.*

Proof. By Theorem 3.1.7, there exists a bounded function h_1 on F such that $h_1 = f_1$ on $F \backslash E$ and h_1 is harmonic at each vertex of E. Choose a function s on T such that $(-\Delta)s = h_1$ on F (Theorem 5.1.5). Again by Theorem 3.1.7, there exists a bounded function g on F such that $g = f_2 - s$ on $F \backslash E$ and $\Delta g = 0$ at each vertex in E. Take now $h_2 = s + g$ on F. Then $h_2 = f_2$ on $F \backslash E$ and $(-\Delta)h_2 = h_1$ at each vertex on E.

Repeated applications of this method produce a function $h = (h_i)_{m \geq i \geq 1}$ on F such that $h_i = f_i$ on $F \backslash E$ and $(-\Delta)h_j(x) = h_{j-1}(x)$ for each vertex $x \in E$, for $2 \leq j \leq m$. Since $(-\Delta)h_1(x) = 0$ if $x \in E$, we conclude that h is an m-harmonic function at each vertex of E.

As for the uniqueness of h, suppose $u = (u_i)_{m \geq i \geq 1}$ is another solution defined on F such that $u_i = f_i$ on $F \backslash E$ and u is m-harmonic at each vertex of E. Let $u - h = v = (v_i)_{m \geq i \geq 1}$. Then $(-\Delta)v_1 = 0$ at each vertex of E and $v_1 = 0$ on $F \backslash E$. Hence by the maximum principle (Property 7 of superharmonic functions, Sect. 3.1), $v_1 \equiv 0$. Consequently, v_2 is harmonic at each vertex of E and $v_2 = 0$ on $F \backslash E$ and hence $v_2 \equiv 0$. Proceeding thus, we prove that $v \equiv 0$. □

Limit of a sequence of m-superharmonic functions. We have used the fact that if a sequence of superharmonic functions on a network converges to a finite limit at each vertex of T, then the limit also is superharmonic. A similar result for m-superharmonic functions is given in the following theorem. It is a discrete analogue of a result concerning the limit of a sequence of polyharmonic functions on a Dirichlet domain [57, p.25], similar to an earlier result due to Montel for a sequence of harmonic functions.

Theorem 5.1.14. *Let $v^n = (v_i^n)_{m \geq i \geq 1}$ be a sequence of m-superharmonic (respectively m-harmonic) functions on T. Suppose $v_m^n(x)$ converges to a finite limit $v_m(x)$ as $n \to \infty$ for every x in T. Then the sequence $\{v^n\}$ converges to an m-superharmonic (respectively m-harmonic) function $v = (v_i)_{m \geq i \geq 1}$ on T.*

Proof.

$$\begin{aligned}(-\Delta)v_m^n(x) &= v_m^n(x) - \sum_{y\sim x} p(x,y)v_m^n(y)\\ &\to v_m(x) - \sum_{y\sim x} p(x,y)v_m(y) \text{ as } n\to\infty\\ &= (-\Delta)v_m(x).\end{aligned}$$

Since $v^n = (v_i^n)_{m\geq i\geq 1}$ is m-superharmonic, $(-\Delta)v_m^n = v_{m-1}^n$ so that we have proved that $v_{m-1}^n(x)$ converges to a finite limit $(-\Delta)v_m(x)$. Write $v_{m-1}(x) = \lim_{n\to\infty} v_{m-1}^n(x) = (-\Delta)v_m(x)$.

Proceeding similarly, we show that $v_i^n(x)$ converges to $(-\Delta)v_{i+1}(x) = v_i(x)$ for $1\leq i\leq m-1$. Now, for every n, v_1^n is superharmonic (respectively harmonic) on T. Hence, $v_1 = \lim_{n\to\infty} v_1^n$ is superharmonic (respectively harmonic) on T. Consequently, $v = (v_i)_{m\geq i\geq 1}$ is an m-superharmonic (respectively m-harmonic) function on T such that $v^n \to v$. □

5.2 Polyharmonic Functions with Point Singularity

Remark 2 following Corollary 5.1.3 states that for any y in T, there exists a superharmonic function q_1 on T such that q_1 is harmonic at each vertex in T except at y. Let $q = (q_i)_{m\geq i\geq 1}$ be an m-superharmonic function generated by q_1 on T. Then, q is an m-superharmonic function on T such that q is m-harmonic at each vertex of T except at y. This function q is not always positive on T. For example, in the Euclidean space $\Re^4$, if we take $q_1(x) = |x|^{-2}$, then $q(x) = \left(h(x) - \frac{1}{2}\log|x|, |x|^{-2}\right)$ is 2-superharmonic generated by q_1, where $h(x)$ is harmonic on $\Re^4$. However, there is no harmonic function h on $\Re^4$ to make $h(x) - \frac{1}{2}\log|x| > 0$. For, $h(x) > \frac{1}{2}\log|x|$ implies that h is positive on $\Re^4$ and hence a constant. But in $\Re^5$, if we take $q_1(x) = |x|^{-3}$, then $q(x) = \left(\frac{1}{2}|x|^{-1}, |x|^{-3}\right)$ is a positive 2-superharmonic function generated by q_1. Actually, positive 2-superharmonic functions with point biharmonic singularity can be determined on $\Re^n$ only if $n\geq 5$. In this section, we take up the problem of finding sufficient conditions for T to possess positive m-superharmonic functions.

A polyharmonic function $u\geq 0$ of order m defined on an open set D in $\Re^n$, $n\geq 2$, is said to be *completely superharmonic* if $(-\Delta)^k u\geq 0$ for $1\leq k < m$. If D is a star domain with centre 0 and if

$$u(x) = |x|^{2m-2} h_m(x) + |x|^{2m-4} h_{m-1}(x) + ... + |x|^2 h_2(x) + h_1(x)$$

is the Almansi representation of a completely superharmonic function u of order m on D, then the harmonic functions $h_m, h_{m-1}, ..., h_2, h_1$ are alternatively positive and negative in D [57, p.17].

Let now $s > 0$ be a completely superharmonic function of order m in a domain D in $\Re^n, n \geq 2$. Let us write $s = (s_i)_{m \geq i \geq 1}$ as in the previous section, so that every s_i is positive superharmonic and s_1 is positive harmonic. Take $u_1 = (\hat{R}^E_{s_1})_D$ where E is a non-polar compact set contained in D (that is, u_1 is the balayée of s_1 in D, Brelot [27, p.34]). Then $u_1 > 0$ is superharmonic and $u_1 \leq s_1$ in D and as in Theorem 5.1.10, there exists a positive m-superharmonic function $u = (u_i)_{m \geq i \geq 1}$ on D such that each u_i is a potential and $u_i \leq s_i$ for $1 \leq i \leq m$. Now u is a very special type of m-superharmonic function in which each component is a positive potential. This shows that for the existence of positive completely superharmonic functions of order m in a domain D in $\Re^n$, there should be some relation between m and n.

In fact, there are no positive completely superharmonic function of order $m > 1$ on the whole of $\Re^n$. For in $\Re^n$, $n \geq 3$, if $\mu \geq 0$ is a Radon measure such that $\int\limits_{y \in \Re^n} \frac{1}{|x-y|^{n-2}} d\mu(y)$ is finite at some x (for example, $x = 0$), then $p(x) = \int\limits_{y \in \Re^n} \frac{1}{|x-y|^{n-2}} d\mu(y)$ is a positive potential on $\Re^n$. Actually for $p(x)$ to be a potential, it is necessary and sufficient that $\int\limits_{|y|>1} \frac{d\mu(y)}{|y|^{n-2}} < \infty$. As a consequence, there cannot be any positive superharmonic function s on $\Re^n, n \geq 2$, such that $(-\Delta)s = 1$. For, when $n = 2$, any positive superharmonic function s on $\Re^2$ is a constant and hence $(-\Delta)s = 0$; when $n \geq 3$, if s is a positive superharmonic function, then s is the sum of a potential p and a non-negative harmonic function h. Since $(-\Delta)s = 1$ by assumption, we find $(-\Delta)p = 1$. Now if $(-\Delta)p(y)dy = d\mu(y)$, then [27, p.47] $p(x) = \varphi_n \int\limits_{y \in \Re^n} \frac{d\mu(y)}{|x-y|^{n-2}}$ for a suitable constant $\varphi_n > 0$, so that $\int\limits_{|y|>1} \frac{d\mu(y)}{|y|^{n-2}}$ should be finite. This means that $\int\limits_{|y|>1} \frac{(-\Delta)p(y)}{|y|^{n-2}} dy$ is finite. Since $(-\Delta)p = 1$, we should find $\int\limits_{|y|>1} \frac{dy}{|y|^{n-2}}$ finite. However,

$$\int\limits_{|y|>1} \frac{dy}{|y|^{n-2}} = \int\limits_{1}^{\infty} \int\limits_{S} \frac{1}{r^{n-2}} r^{n-1} dr d\sigma,$$

where $d\sigma$ is the surface area on the unit sphere S. This leads to a contradiction that $\int\limits_1^\infty r dr$ is finite. Hence, for a positive superharmonic function s on $\Re^n, n \geq 2$, $(-\Delta)s = 1$ is not possible.

Suppose now $u = (u_i)_{m \geq i \geq 1}$ is a positive completely superharmonic function of order $m > 1$ on $\Re^n$. Then $u_1 \geq 0$ is harmonic on $\Re^n$, so that u_1 is a positive constant c. Then, $(-\Delta)u_2 = c$ implies that u_2 is superharmonic on $\Re^n$ and by assumption $u_2 > 0$. Thus there exists a positive superharmonic function u_2 such that $(-\Delta)u_2 = c$. We have just seen that this is not possible. Consequently, there cannot be any positive completely superharmonic function of order $m > 1$ on $\Re^n$.

But positive m-superharmonic functions exist on $\Re^n$ if $n \geq 2m+1$. For example, if $u_m(x) = |x|^{2m-n}$, then u_m together with its derivatives $(-\Delta)^i u_m$ defines a positive m-superharmonic function on $\Re^n$. To study the discrete analogue of this situation, we introduce the following definition.

Definition 5.2.1. An m-superharmonic function $s \geq 0$ on a subset A in T is said to be a polypotential of order m (an m-potential, for short) if the g.m-h.m of s on A is 0.

Theorem 5.2.1. *If s is an m-superharmonic function on a set A in T, majorizing an m-subharmonic function on A (in particular, if s is positive), then s is the unique sum of an m-potential and an m-harmonic function on A.*

Proof. Since s has an m-subharmonic minorant on A, let h be the g.m-h.m. of s on A (Theorem 5.1.9). Then, $p = s - h$ is a positive m-superharmonic function on A and its g.m-h.m. on A is 0. Hence, p is an m-potential on A. As for the uniqueness of representation, suppose $s = p^{'} + h^{'}$ is another such representation. Then, $p \geq h^{'} - h$ so that $h^{'} - h \leq 0$. Similarly, $h - h^{'} \leq 0$, so that $h = h^{'}$ and consequently, $p = p^{'}$. □

Theorem 5.2.2. *Let $s = (s_i)_{m \geq i \geq 1} \geq 0$ be an m-superharmonic function on A. Then, s is an m-potential if and only if every s_i is a potential on A.*

Proof. Let s be an m-potential on A. Suppose s_j is not a potential for some j, $m \geq j \geq 1$. Let h_j be the g.h.m. of s_j in A. Then, as in Theorem 5.1.9, we can construct an m-harmonic minorant $h = (h_i)_{m \geq i \geq 1}$ of s on A such that $h_i = 0$ if $i < j$. Since h is not the zero function, s cannot be an m-potential, a contradiction.

Conversely, suppose every s_i is a potential on A. Let $h = (h_i)_{m \geq i \geq 1}$ be the g.m-h.m. of s on A. Since $0 \leq h_1 \leq s_1$ on A, $h_1 \equiv 0$. Since $(-\Delta)h_2 = h_1$, we conclude that is h_2 harmonic on A. Again, $0 \leq h_2 \leq s_2$ and s_2 is a potential on A lead to the conclusion $h_2 \equiv 0$. Proceeding similarly, we show that $h \equiv 0$. Hence, s is an m-potential on A. □

Corollary 5.2.3. *Let $s = (s_i)_{m \geq i \geq 1}$ be an m-superharmonic function on a subset A in T. Suppose q is a potential on A such that for each i, $|s_i| \leq q$ outside a finite set in A. Then, s is an m-potential on A.*

Proof. First note that if u is a superharmonic function on A and $|u| \leq p$ outside a finite set F, where p is a potential on A, then u is a potential on A. For, F being a finite set, we can find a potential majorizing $|u|$ on F. Hence, we can assume that $|u| \leq p$ on A. Since the superharmonic function u majorizes the subharmonic function $-p$ on A, $u = p_1 + h$ where p_1 is a potential and h is harmonic on A. Now, the subharmonic function $|h| \leq p + p_1$ on A. Consequently, $h \equiv 0$ and u is the potential p_1 on A.

In this corollary, since s_1 is superharmonic on A such that $|s_1| \leq q$ outside a finite set in A, s_1 should be a potential on A. Since $(-\Delta)s_2 = s_1 \geq 0$ on A, s_2 is a superharmonic function on A. By hypothesis, $|s_2| \leq q$ outside a finite set in A and hence s_2 is a potential on A. A similar procedure shows that s_i is a potential for each i, $m \geq i \geq 1$. Hence, by the above theorem, $s = (s_i)_{m \geq i \geq 1}$ is an m-potential on A. □

In a tree, there may or may not be any positive m-potential. This is the case even in the classical potential theory on the Euclidean spaces. For example, when $m = 2$, we have a positive bi-potential (p_2, p_1) on $\Re^n$ if and only if $n \geq 5$. Since $(-\Delta) |x|^{4-n} = c_n |x|^{2-n}$, for some $c_n > 0$ and $n \geq 5$, $\left(|x|^{4-n}, c_n |x|^{2-n} \right)$ is a bi-potential. *We shall say that T is an m-potential tree if there exists a positive m-potential on T.* If we know that there exists at least one m-potential on T, then we can construct many m-potentials with varied properties.

Theorem 5.2.4. *Let $Q = (Q_i)_{m \geq i \geq 1}$ be a positive m-potential on a subset A in T. Let p_1 be a positive potential on A such that $p_1 \leq Q_1$. Then there exists a unique m-potential $p = (p_i)_{m \geq i \geq 1}$ generated by p_1 such that $p_i \leq Q_i$ for $m \geq i \geq 1$.*

Proof. Choose a function s_1 on T such that $(-\Delta) s_1 = p_1$ on A. By hypothesis, $(-\Delta) Q_2 = Q_1 \geq p_1$ on $\overset{0}{A}$. Choose a superharmonic function t_1 on T such that $(-\Delta) t_1 = Q_1 - p_1$ on $\overset{0}{A}$. Then, $Q_2 = s_1 + t_1 +$ (a harmonic function h_1) on A.

Since $Q_2 \geq 0$, the superharmonic function s_1 has a subharmonic minorant on A, so that $s_1 = p_2 + h_2$ on A where p_2 is a potential and h_2 is harmonic on A; similarly, $t_1 = p_2^{'} + h_2^{'}$ on A. Thus, $Q_2 = (p_2 + p_2^{'}) + (h_2 + h_2^{'} + h_1)$ on A. Equating the potential parts, we have $Q_2 = p_2 + p_2^{'}$. Hence, $p_2 \leq Q_2$; note that $(-\Delta) p_2 = (-\Delta) s_1 = p_1$ on $\overset{0}{A}$.

Proceeding in a similar manner, we construct potentials p_i on A such that $p_i \leq Q_i$ for every i, $m \geq i \geq 1$, and that $p = (p_i)_{m \geq i \geq 1}$ is an m-potential on A. As for the uniqueness, suppose $(q_m, q_{m-1}, ..., q_2, p_1)$ is another m-potential generated by p_1 on A. Since $(-\Delta) p_2 = p_1 = (-\Delta) q_2$ on $\overset{0}{A}$, we conclude that $p_2 = q_2 +$ (a harmonic function) on A. Hence, $p_2 = q_2$ since p_2 and q_2 are potentials on A. A similar procedure leads to the conclusion that p_1 generates a unique potential on A. □

Corollary 5.2.5. *If T is an m-potential tree, then any potential p_1 with finite harmonic support generates a unique m-potential on T.*

Proof. Let $Q = (Q_i)_{m \geq i \geq 1}$ be an m-potential on T. Since p_1 has finite harmonic support, $p_1 \leq \alpha Q_1$ on T for some $\alpha > 0$, by the Domination Principle (Theorem 3.3.6); and $\alpha Q = (\alpha Q_i)_{m \geq i \geq 1}$ is an m-potential on T. Hence, by the above theorem, p_1 generates an m-potential on T. □

m-harmonic Green function. Let T be an m-potential tree. For a given vertex y, let $G_y(x)$ denote the Green function on T, with point harmonic support at $\{y\}$. Then, by Corollary 5.2.5, $G_1(x, y) = G_y(x)$ generates a unique m-potential

$$G_y^{(m)}(x) = G^{(m)}(x, y) = (G_m(x, y), ..., G_2(x, y), G_1(x, y))$$

on T which we shall refer to as the *m-harmonic Green function* of T with point support $\{y\}$. We write $(-\Delta)^m G_y^{(m)}(x) = \delta_y(x)$. Since $G_2(x,y)$, as a function of x is a potential on T when y is fixed, it has a representation (Theorem 3.3.1)

$$G_2(x,y) = \sum_{z\in T}(-\Delta)G_2(z,y)G_1(x,z)$$

where the Laplacian is taken in the first variable. Hence,

$$G_2(x,y) = \sum_{z\in T} G_1(z,y)G_1(x,z).$$

Similarly, $G_i(x,y) = \sum\limits_{z_1,\ldots,z_{i-1}\in T} G_1(z_{i-1},y)G_1(z_{i-2},z_{i-1})\ldots G_1(x,z_1)$, for $m \geq i \geq 1$.

Proposition 5.2.6. *If a positive potential p_1 in T generates an m-potential $p = (p_i)_{m\geq i\geq 1}$, then $p_i(x) = \sum\limits_{z\in T}(-\Delta)p_1(z)G_i(x,z)$, for all i, $m \geq i \geq 1$.*

Proof. Since p_2 is a potential and $(-\Delta)p_2 = p_1$, we write

$$\begin{aligned}
p_2(x) &= \sum_{z_1\in T}(-\Delta)p_2(z_1)G_1(x,z_1) \\
&= \sum_{z_1\in T} p_1(z_1)G_1(x,z_1) \\
&= \sum_{z_1\in T}\left[\sum_{z_2\in T}(-\Delta)p_1(z_2)G_1(z_1,z_2)\right]G_1(x,z_1) \\
&= \sum_{z_2}(-\Delta)p_1(z_2)\left[\sum_{z_1}G_1(z_1,z_2)G_1(x,z_1)\right] \\
&= \sum_{z_2\in T}(-\Delta)p_1(z_2)G_2(x,z_2) \\
&= \sum_{z\in T}(-\Delta)p_1(z)G_2(x,z).
\end{aligned}$$

Similarly, $p_i(x) = \sum_{z\in T}(-\Delta)p_1(z)G_i(x,z)$, for all i, $m \geq i \geq 1$. □

Theorem 5.2.7. *Assume there are positive potentials on T. Let $G_1(x,y)$ denote the Green function on T such that $(-\Delta)G_1(x,y) = \delta_y(x)$. Then T is an m-potential tree if and only if for a pair (and hence for every pair) of vertices u and v in T,*

$$G_m(u,v) = \sum_{z_1,z_2,..,z_{m-1}} G_1(u,z_1)G_1(z_1,z_2)\ldots G_1(z_{m-1},v) < \infty.$$

Proof. If the above sum is finite, take $u = z_1 = z_2 = ... = z_{m-2}$. Since $G_1(u,u) < \infty$, we find that $\sum_{z_{m-1}} G_1(u, z_{m-1})G_1(z_{m-1}, v) < \infty$. Hence, $q_2(x) = \sum_{z_{m-1}} G_1(x, z_{m-1})G_1(z_{m-1}, v)$ is a potential on T such that $(-\Delta)q_2(x) = G_1(x, v)$. Similarly, since

$$\sum_{z_{m-2},z_{m-1}} G_1(u, z_{m-2})G_1(z_{m-2}, z_{m-1})G_1(z_{m-1}, v) < \infty,$$

we conclude that $\sum_{z_{m-2}} G_1(u, z_{m-2})q_2(z_{m-2}) < \infty$. Hence, $q_3(x) = \sum_{z_{m-2}} G_1(x, z_{m-2})q_2(z_{m-2})$ is a potential and $(-\Delta)q_3(x) = q_2(x)$. Proceeding similarly, we construct the potential q_j for which $(-\Delta)q_j = q_{j-1}$, $2 \leq j \leq m$, where $q_1(x) = G_1(x, v)$. This means that $q = (q_i)_{m \geq i \geq 1}$ is an m-potential on T and hence T is an m-potential tree.

Conversely, if T is an m-potential tree, then the m-harmonic Green function $G_y^{(m)}(x) = (G_m(x, y), ..., G_2(x, y), G_1(x, y))$ exists on T. Here,

$$G_m(x, y) = \sum_{z_1,z_2,...,z_{m-1}} G_1(x, z_1)G_1(z_1, z_2)...G_1(z_{m-1}, y)$$

is a potential on T and hence is finite for every pair of vertices x and y. □

Proposition 5.2.8. *If T is a standard homogeneous tree where each vertex has degree $q + 1, q \geq 2$, and the transition probabilities are all $(q + 1)^{-1}$, then T is a bi-potential tree.*

Proof. Let T be a standard homogeneous tree of degree $q + 1$, $q \geq 2$; that is, every vertex in T has $q + 1$ neighbours [31, p.262 and p.212]. Let $d(s, t)$ be the length of the (geodesic) path connecting s to T. Then, $d(s, t) = 1$ if and only if $s \sim t$ and $d(s, s_2) \equiv d(s, s_1) + d(s_1, s_2)(mod\, 2)$. There exist positive potentials on T and the Green function on T is $G(s, t) = \frac{q}{q-1} \cdot \frac{1}{q^{d(s,t)}}$. Let us fix two vertices u and $v \sim u$. Then, for $x \in T$,

$$G(u, x)G(x, v) = \frac{q^2}{(q-1)^2} \cdot \frac{1}{q^{d(u,x)+d(x,v)}}.$$

Since $d(u, x)+d(x, v) \equiv 1(mod\, 2)$, if we write $A_n = \{x : d(u, x) + d(x, v) = n\}$, then n is odd and $T = \cup A_n$. Now, $|d(u, x) - d(x, v)| \leq 1$, and $d(u, x) = d(x, v) \pm 1$, so that if $x \in A_n$, then $d(u, x) = \frac{n-1}{2}$ or $d(x, v) = \frac{n-1}{2}$. This implies that *card* $\{A_n\} \leq 2(q + 1)^{\frac{n-1}{2}}$. Hence,

$$\sum_{x \in A_n} G(u, x)G(x, v) \leq 2(q + 1)^{\frac{n-1}{2}} \cdot \frac{q^2}{(q-1)^2} \cdot \frac{1}{q^n} = \lambda \frac{(q + 1)^{\frac{n}{2}}}{q^n},$$

where $\lambda = 2(q+1)^{-\frac{1}{2}} \cdot \frac{q^2}{(q-1)^2}$. Consequently, $\sum_{x \in T} G(u,x)G(x,v) \leq \lambda \sum_n \frac{(q+1)^{\frac{n}{2}}}{q^n}$. This last series converges since $q \geq 2$. Hence, by Theorem 5.2.7, T is a bi-potential tree. □

Theorem 5.2.9. *T is an m-potential tree if and only if given an m-harmonic function $h = (h_i)_{m \geq i \geq 1}$ outside a finite set, there exists a (unique) m-harmonic function $H = (H_i)_{m \geq i \geq 1}$ on T such that for some potential q on T, $|h_i - H_i| \leq q$ outside a finite set for each i.*

Proof. Suppose T is an m-potential tree. Since h_1 is harmonic outside a finite set, there exists a harmonic function H_1 on T and a potential p_1 with finite harmonic support on T such that $|h_1 - H_1| \leq p_1$ outside a finite set (Corollary 3.2.7). Since p_1 has finite harmonic support, it generates an m-potential $p = (p_i)_{m \geq i \geq 1}$ on T. Let $(-\Delta)H_2' = H_1$ on T. Now, by the assumption$(-\Delta)h_2 = h_1$ outside a finite set, so that $\left|(-\Delta)h_2 - (-\Delta)H_2'\right| \leq p_1 = (-\Delta)p_2$ outside a finite set A in T.

Write $s = h_2 - H_2' + p_2$ and $t = h_2 - H_2' - p_2$. Then, on $T \backslash A$, s is superharmonic and t is subharmonic such that $t \leq s$. Hence, there exists a harmonic function h_0 outside a finite set such that $t \leq h_0 \leq s$. Then, as before we can find a harmonic function u on T and a potential v with finite harmonic support in T such that $|h_0 - u| \leq v$ outside a finite set. Set $H_2 = H_2' + u$. Note $(-\Delta)H_2 = (-\Delta)H_2' = H_1$ on T and $|h_2 - H_2| = \left|(h_2 - H_2' - h_0) - (u - h_0)\right| \leq p_2 + v$ outside a finite set.

Now, since v is a potential with finite harmonic support, there exists a potential v_1 on T such that $(-\Delta)v_1 = v$ on T. Hence, if $q_3 = p_3 + v_1$, then $(-\Delta)q_3 = p_2 + v$. Set $q_2 = p_2 + v$. Thus far, we have proved that there exist H_2 such that $(-\Delta)H_2 = H_1$ and the potential q_3 such that $(-\Delta)q_3 = q_2$ and $|h_2 - H_2| \leq q_2$ outside a finite set in T.

Proceeding similarly we prove that for $m \geq j \geq 3$ also, there exist H_j on T such that $(-\Delta)H_j = H_{j-1}$ and a potential q_j such that $|h_j - H_j| \leq q_j$ outside a finite set in T. Write $q = p_1 + q_2 + ... + q_m$. Then, $H = (H_i)_{m \geq i \geq 1}$ is an m-harmonic function on T such that $|h_i - H_i| \leq q$ outside a finite set for each i, $m \geq i \geq 1$.

As for the uniqueness of the m-harmonic function H, suppose there is another m-harmonic function $H' = (H_i')_{m \geq i \geq 1}$ and a potential q' on T such that $\left|h_i - H_i'\right| \leq q'$ outside a finite set in T. Then, $(H_1 - H_1')$ is harmonic on T and $\left|H_1 - H_1'\right| \leq q + q'$ outside a finite set. Hence $H_1 - H_1' \equiv 0$. Then, $(-\Delta)(H_2 - H_2') = H_1 - H_1' \equiv 0$, so that $(H_2 - H_2')$ is harmonic on T. But $\left|H_2 - H_2'\right| \leq q + q'$ outside a finite set. Hence $H_2 - H_2' \equiv 0$. This procedure establishes $H_i = H_i'$ for all i, $m \geq i \geq 1$.

To prove the converse, first note that the assumption in the theorem implies that there are positive potentials on T. Choose a potential p_1 in T with finite harmonic support A. Let $p = (p_i)_{m \geq i \geq 1}$ be an m-superharmonic function generated by p_1. This should be m-harmonic outside A. Hence, by the assumption there exists an

m-harmonic function $H = (H_i)_{m \geq i \geq 1}$ on T and a potential q such that $|p_i - H_i| \leq q$ outside a finite set. Set $s_i = p_i - H_i$ on T. Then, $s = (s_i)_{m \geq i \geq 1}$ is an m-superharmonic function on T and $|s_i| \leq q$ outside a finite set for each i. This implies that s is an m-potential on T (Corollary 5.2.3). □

If $s = (s_i)_{m \geq i \geq 1}$ is an m-superharmonic function on T and if A is the harmonic support of s_1, then we say that the m-harmonic support of s is A.

Theorem 5.2.10. *(Domination Principle for m-potentials) Let $s = (s_i)_{m \geq i \geq 1}$ be a positive m-superharmonic function and $p = (p_i)_{m \geq i \geq 1}$ be an m-potential on T. Let A be a subset of T containing the m-harmonic support of p. Suppose $s_1 \geq p_1$ on A. Then, $s \geq p$ on T.*

Proof. By Theorem 3.3.6, the fact that $s_1 \geq p_1$ on A implies that $s_1 \geq p_1$ on T. Let $(-\Delta)u = s_1 - p_1 = (-\Delta)s_2 - (-\Delta)p_2$. Then, u is superharmonic on T and $u + p_2 +$ (a harmonic function)$= s_2 \geq 0$.

Since u has a subharmonic minorant on T, it is the unique sum of a potential q and a harmonic function on T. Hence, $s_2 = q + p_2 +$ (a harmonic function) on T.

Use now the uniqueness of decomposition of s_2 as the sum of a potential and a non-negative harmonic function to conclude that $s_2 \geq q + p_2 \geq p_2$. A similar procedure leads to the conclusion $s_i \geq p_i$ for each i, $m \geq i \geq 1$. □

Theorem 5.2.11. *(Balayage of m-potentials) Let $p = (p_i)_{m \geq i \geq 1}$ be an m-potential on T. Let A be a subset of T. Then, there exists an m-potential R_p^A on T such that $R_p^A \leq p$ on T, $R_p^A = p+$ (an $(m-1)$-harmonic function) at each vertex of A and R_p^A is m-harmonic at each vertex on $T \backslash A$.*

Proof. By Theorem 3.1.10, there exists a potential q_1 on T such that $q_1 \leq p_1$ on T, $q_1 = p_1$ on A and $(-\Delta)q_1(x) = 0$ at each vertex $x \in T \backslash A$. Then, q_1 generates an m-potential $R_p^A = q = (q_i)_{m \geq i \geq 1}$ such that $q \leq p$ on T (Theorem 5.2.4). Since q_1 is harmonic at each vertex of $T \backslash A$, the m-potential q is m-harmonic at each vertex of $T \backslash A$.

Further $q_1 = p_1$ on A so that $(-\Delta)q_2 = q_1 = p_1 = (-\Delta)p_2$ and hence $q_2 = p_2 + h_2$ on A where h_2 is harmonic at each vertex of A. Let H_3 be a function defined on T such that $(-\Delta)H_3 = h_2$ on A, so that $(-\Delta)q_3 = (-\Delta)p_3 + (-\Delta)H_3$ on A, and $q_3 = p_3 + H_3 + u$ on A where u is harmonic at each vertex of A. Write $h_3 = H_3 + u$ on A. Then, $q_3 = p_3 + h_3$ on A and $(-\Delta)h_3 = h_2$ on A. A similar procedure leads to the construction of $(h_m, h_{m-1}, ..., h_2)$ which is $(m-1)$-harmonic at each vertex of A; this can be identified with the m-harmonic function $h = (h_m, h_{m-1}, ..., h_2, 0)$ so that $q = p + h$ on A, where h is $(m-1)$-harmonic at each vertex of A. □

Remark 5.2.1. Let $p = (p_i)_{m \geq i \geq 1}$ be an m-potential on T and A be a subset of T. Write as above $R_p^A = q = (q_i)_{m \geq i \geq 1}$. Let $\Im$ be the family of all positive m-superharmonic functions $s = (s_i)_{m \geq i \geq 1}$ on T such that $s_1 \geq p_1$ on A; consequently $s_1 \geq q_1$ on A which contains the harmonic support of q_1. Then, by Theorem 5.2.10, $s \geq q$ on T. Thus, if $s \in \Im$, then $s \geq q$ on T. Note that $q \in \Im$. Hence, $\inf\limits_{s \in \Im} s = q = R_p^A$.

5.3 Riesz-Martin Representation for Positive m-Superharmonic Functions

A domain Ω in the Euclidean space $\Re^n, n \geq 2$, is called a Green domain if the Green function $G(x, y)$ is well-defined on Ω. Consequently, any domain in $\Re^n, n \geq 3$, is a Green domain and in $\Re^2$, Ω is a Green domain if and only if $\Re^2 \backslash \Omega$ is not locally polar. (A set E in $\Re^2$ is said to be locally polar if and only if there exists a superharmonic function s on $\Re^2$ such that $e \subset \{x : s(x) = \infty\}$). In Brelot [29, pp.111–115], we have a method to construct a compactification $\bar{\Omega}$ (called the Martin compactification) of Ω with the following properties: $\bar{\Omega}$ is a compact set in which Ω is dense open. If y_0 is a point fixed in Ω and if $G(x, y)$ is the Green function, then write

$$k_y(x) = k(x, y) = \frac{G(x, y)}{G(x, y_0)}$$

with the convention $k(y_0, y_0) = 1$. Then, $\bar{\Omega}$ is the unique metrisable compactification (up to a homeomorphism) of Ω such that:

1. Ω is dense open in the compact space $\bar{\Omega}$;
2. $k_y(x)$, $y \in \Omega$, extends as a continuous function of x on $\bar{\Omega}$;
3. the family of these extended continuous functions on $\bar{\Omega}$ separates the points $X \in \Lambda = \bar{\Omega} \backslash \Omega$.

$\bar{\Omega}$ is called the *Martin compactification* of Ω and $\Lambda = \bar{\Omega} \backslash \Omega$ is called the *Martin boundary*. A non-negative harmonic function u on Ω is called *minimal* if and only if for any harmonic function v in Ω, $0 \leq v \leq u$, we have $v = \alpha u$ for some constant α, $0 \leq \alpha \leq 1$.

It can be proved that every minimal harmonic function $u(y)$ in Ω is of the form $u(y_0)k(y, X)$ for some $X \in \Lambda$. The points $X \in \Lambda$ corresponding to these minimal harmonic functions are called the *minimal points* of Λ, and the set of minimal points of Λ is denoted by Λ_1, called the *minimal boundary*. With these preliminaries, we can state the *Martin representation* theorem: *Let* $u \geq 0$ *be a harmonic function on* Ω. *Then, there exists a unique Radon measure* $\mu \geq 0$ *on* Λ *with support in the minimal boundary* $\Lambda_1 \subset \Lambda$ *such that* $u(y) = \int\limits_{X \in \Lambda_1} k(y, X) d\mu(X)$, *for* $y \in \Omega$.

In the particular case of $\Omega = B(0, 1)$, the unit ball in $\Re^n, n \geq 2$, taking the fixed point y_0 as the centre 0, we find that the Martin boundary $\Lambda = \bar{\Omega} \backslash \Omega$ is homeomorphic to the unit sphere S and $k(y, X)$ is the Poisson kernel; also, $\Lambda_1 = \Lambda = S$. Consequently, the Martin representation is the well-known result [13, p. 105]: *If* u *is positive harmonic on* B, *then there exists a unique non-negative regular Borel measure* μ *on* S *such that* $u(x) = \int\limits_S P(x, X) d\mu(X)$, *where* $P(x, X)$, $x \in B$, $X \in S$ *is the Poisson kernel.*

It is easy to obtain a generalisation of the Martin representation for positive harmonic functions on Ω to positive superharmonic functions on Ω. Suppose s is a positive superharmonic function on a Green domain Ω in $\Re^n, n \geq 2$. Then, by the Riesz representation theorem, s is the unique sum of a potential p and a non-negative harmonic function h on Ω. Now, p can be represented in the form of an integral $C \int_{y\in\Omega} G(x,y)(-\Delta p)(y)dy$, where $C > 0$ is a constant and the Radon measure $(-\Delta)p(y)dy$ is calculated in the sense of distributions [27, p. 47]. Since $(-\Delta)p \geq 0$, there exists a Radon measure $\nu = C(-\Delta p)$. As for the non-negative harmonic function h, we can use the Martin representation. Consequently, we have the following Riesz-Martin representation for positive superharmonic functions: *Let $s \geq 0$ be a superharmonic function defined on a Green domain Ω in $\Re^n, n \geq 2$. Then, there exist two non-negative Radon measures ν on Ω and μ on Λ with support in Λ_1 such that for $x \in \Omega$,*

$$s(x) = \int_{y\in\Omega} G(x,y)d\nu(y) + \int_{X\in\Lambda_1} k(x,X)d\mu(X).$$

For a deep study of Martin boundary in the context of probability theory, see Doob [43], and Meyer [54]. Cartier [31, pp.235–237] shows how this Martin representation for positive harmonic functions can be presented in the context of an infinite tree. Let T be an infinite tree with the Green function $G(x,y)$. Two paths in T are said to be *equivalent paths* if they have an infinite number of common vertices. With this equivalence relation, an equivalence class of infinite paths is called an *end*. Let B denote the set of all ends in T. Write $\hat{T} = T \cup B$. Then, [31, Theorem 1.1, p. 220] a topology can be defined on $\hat{T}$ so that $\hat{T}$ is compact, metrisable and locally discontinuous; B is compact, metrisable; and T (with its discrete topology) is dense open in $\hat{T}$.

Fix a vertex y_0 in T. Let $k_y(x) = \frac{G(y,x)}{G(y_0,x)}$, for x, y in T. Then, $k_y(x)$ can be extended as a locally constant (and hence a continuous) function for $x \in \hat{T}$ such that $k_{y_0}(x) = 1$ (normalisation) and $k_y(x) > 0$ for all $x \in \hat{T}$; moreover, for any $X \in B$, $k_y(X)$ is harmonic on T. The compact metrisable space B is the *Martin boundary* of T. Further,

1. If $h(y)$ is a positive harmonic function on T, then there exists a unique Borel measure $\mu \geq 0$ on B such that $h(y) = \int_B k_y(X)d\mu(X)$.
2. If $s(y)$ is a positive superharmonic function on T, then there exist two unique measures $\nu \geq 0$ on T and $\mu \geq 0$ on B such that $s(y) = \sum_{x\in T} G(y,x)\nu(x) + \int_B k_y(X)d\mu(X)$.

Thus, a positive superharmonic function on T is determined by two positive measures, one defined on T and the other on B. We shall show now that a positive

m-superharmonic function on an m-potential tree T is determined by $(m+1)$ uniquely defined positive measures, one on T and the other m measures on B. For that, we need a Riesz-Martin representation for positive m-superharmonic functions on an m-potential tree. First, we start with a Riesz representation for positive m-superharmonic functions on an m-potential tree. From Theorem 5.1.10, we have: *Let T be an m-potential tree. Let $s = (s_i)_{m \geq i \geq 1}$ be a positive m-superharmonic function on T. Let u_1 be a superharmonic function on T such that $0 < u_1 \leq s_1$. Then, there exists an $(m-1)$-potential $p = (p_j)_{m \geq j \geq 2}$ on T such that $u = (p_m, p_{m-1}, ..., p_2, u_1)$ is a positive m-superharmonic function on T and $0 < u \leq s$ on T. Since p is an $(m-1)$-potential generated by p_2, it is uniquely determined.*

Definition 5.3.1. A positive m-harmonic function $h = (h_i)_{m \geq i \geq 1}$ on T is said to be an m-harmonic potential if each h_j, $m \geq j \geq 2$, is a positive potential on T.

Remark 5.3.1. 1. From the above remark, it follows that if $s = (s_i)_{m \geq i \geq 1}$ is a positive m-superharmonic function on T and if h is a harmonic function on T, $0 < h \leq s_1$, then h generates a unique m-harmonic potential $(p_m, p_{m-1}, ..., p_2, h)$ on T majorized by s.

2. If $h = (h_j)_{k \geq j \geq 1}$ is a k-harmonic potential on T, then by adding $(m-k)$ zeros at the end, we can consider $h = (h_k, h_{k-1}, ..., h_1, 0, 0..., 0)$ as a non-negative m-harmonic function that is a k-harmonic potential on T.

Theorem 5.3.1. *Let $s = (s_i)_{m \geq i \geq 1}$ be a non-negative m-superharmonic function on T. Then there exist an m-potential p and i-harmonic potentials u_i, $m \geq i \geq 1$, such that $s = p + \sum_{i=1}^{m} u_i$ on T. Moreover, this representation is unique.*

Proof. Write $s_1 = p_1 + h_1$, where p_1 is a potential and h_1 is a non-negative harmonic function on T. Since $p_1 \leq s_1$, p_1 generates an m-potential $p = (p_i)_{m \geq i \geq 1}$ and $p \leq s$. Similarly, h_1 generates an m-harmonic potential (if $h_1 > 0$) $u_m = (h_i)_{m \geq i \geq 1}$, $u_m \leq s$ on T. If $h_1 = 0$ at a vertex, then take $u_m \equiv 0$. Let $v = s - p - u_m = (v_m, v_{m-1}, ..., v_2, 0)$. Now, $(-\Delta)s_2 = s_1 = p_1 + h_1 = (-\Delta)p_2 + (-\Delta)h_2$. Hence, $s_2 = p_2 + h_2 +$ (a harmonic function ξ_2) on T.

Since s_2 is a positive superharmonic function and p_2, h_2 are potentials, we conclude $\xi_2 \geq 0$. But $\xi_2 = v_2$; hence v_2 is a non-negative harmonic function on T. Further, for $m \geq j \geq 2$, $(-\Delta)v_j = (-\Delta)s_j - (-\Delta)p_j - (-\Delta)h_j = s_{j-1} - p_{j-1} - h_{j-1} = v_{j-1}$ (taking $v_1 = 0$).

Since $(-\Delta)p_3 + (-\Delta)h_3 = p_2 + h_2 \leq s_2 = (-\Delta)s_3$, we conclude $v_3 = s_3 - p_3 - h_3$ is superharmonic on T. Since $p_3 + h_3 + v_3 = s_3 \geq 0$, we have $-v_3 \leq p_3 + h_3$; here $-v_3$ is subharmonic and $p_3 + h_3$ is a potential, so that $-v_3 \leq 0$. Hence, $p_3 + h_3 \leq s_3$. Then, a similar procedure as above leads to the conclusion $v_4 \geq 0$. Eventually, we show that $v_j \geq 0$ for all j, $m \geq j \geq 2$. Consequently, $(v_j)_{m \geq j \geq 2}$ is a non-negative $(m-1)$-harmonic function on T.

The above procedure starting with v instead of s leads to the unique decomposition $v = u_{m-1} + v'$, where u_{m-1} is an $(m-1)$-harmonic potential and v' is a non-negative $(m-2)$-harmonic function on T. Proceeding in a similar manner, finally we arrive at the unique decomposition

$$s = p + u_m + u_{m-1} + ... + u_1,$$

where p is an m-potential and u_i is an i-harmonic potential (identified with a non-negative m-harmonic function on T). □

In an m-potential tree T, let M be the class of measures $\mu \geq 0$ on T such that $p_1(x) = \sum_{y \in T} G_y(x)\mu(\{y\})$ is a potential that generates an m-potential $(p_i)_{m \geq i \geq 1}$ on T. Let $\Lambda_i, m \geq i \geq 1$, be the class of measures $\nu_i \geq 0$ on the Martin boundary B such that the non-negative harmonic function h_1 on T associated with ν_i [31, p.232] generates an i-harmonic potential u_i on T. We have already remarked that if $u_i = (h_i, h_{i-1}, ...h_1)$ is an i-harmonic potential generated by h_1, then u_i is uniquely determined by h_1. Since h_1 is uniquely determined by a measure ν_i in the class Λ_i, we conclude that an i-harmonic potential u_i is uniquely determined by a measure in the class Λ_i.

Theorem 5.3.2. *A non-negative m-superharmonic function on an m-potential tree T is uniquely determined by $(m+1)$ measures $(\mu, \nu_m, ..., \nu_1) \in M \times \Lambda_m \times ... \times \Lambda_1$.*

Proof. By Theorem 5.3.1, if s is a non-negative m-superharmonic function on an m-potential tree, then s has a unique representation of the form $s = p + \sum_{i=1}^{m} u_i$, where p is an m-potential and u_i is an i-harmonic potential on T. Now, the m-potential $p = (p_i)_{m \geq i \geq 1}$ is determined by p_1 which in turn is uniquely determined by the measure $(-\Delta)p_1 \in M$. Moreover, each u_i is determined by a measure $\nu_i \in \Lambda_i$ as shown above. Consequently, the positive m-superharmonic function s on T is uniquely determined by an element in $M \times \Lambda_m \times ... \times \Lambda_1$. □

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Index

Admissible, 74
Alexandroff point at infinity, 54
Algebraic potential theory, 3
Almansi representation, 110
Axiomatic potential theory, 3
Axiom of proportionality, 66

Bôcher's theorem, 66
Balayage, 21, 30, 62
q-Balayage, 41, 96
Balayage of m-potentials, 128
Barycentres, 60
Base, 61
Biharmonic, 109
Biharmonic classification, 110
Biharmonic Green function, 110
Boundary, 5, 46

Capacitary functions, 62
Capacitary potential, 67
Capacity, 2, 21, 41, 67
Characteristic function, 88
Circled, 5
Classical q-Dirichlet problem, 41
Classical Dirichlet problem, 23, 29
Classical Dirichlet solution for the q-Laplacian, 98
Completely superharmonic, 121
Condenser principle, 3, 29, 42, 62, 99
Conductance, 2, 5
Connected, 5, 46
Convergence theorem, 8
Cycle, 5, 111

Digraph, 1
Dirichlet finite, 2
Dirichlet functions, 2
Dirichlet norm, 2
Dirichlet-Poisson equation, 23, 28
Δ_q-Poisson kernel, 43
Δ_q-Poisson modification, 38, 94
Dirichlet-Poisson problem, 3
Dirichlet-Poisson solution, 37, 62, 69
Dirichlet principle, 33
Dirichlet problem, 21
Dirichlet semi-norm, 32
Dissipation formula, 13
Distance, 5
Domination principle, 21, 26, 62, 65
q-Domination principle, 37, 95
Domination principle for m-potentials, 128
Domination principle for pseudo-potentials, 90
Doob ξ-transform, 103

Edges, 1
Effective resistance, 2, 24
Electric potentials, 3
Electrostatics, 21
End, 130
Energy integral, 2
Equilibrium measures, 23, 35
Equilibrium principle, 2, 21, 27
q-Equilibrium principle, 101
Equivalent paths, 130
Equivalent superharmonic functions, 75
Escape probability, 2, 24
Extremal, 61
Extremal elements, 60
Extremal length, 2

Finite difference approximation, 45
Finite electrical network, 2

Finite section, 52
Flow, 22
Flux at infinity, 78, 79
Flux at infinity of h, 79

Generalised q-Condenser principle, 42
Generalised Dirichlet problem, 50
Generalised Dirichlet solution for the q-Laplacian, 98
Generalised equilibrium principle, 27
Generated by, 115
Graph, 1
Gravitation, 78
Greatest q-harmonic minorant, 40
Greatest harmonic minorant, 47
Greatest m-harmonic minorant, 119
Green domain, 129
Green kernel, 4, 62
Green's formula, 12
Green's function, 2, 30
q-Green's function, 42
q-Green's function for a finite set, 100
Green's function on a set, 58

Harmonic, 7, 46
m-Harmonic, 109, 115
Harmonic dimension of a hyperbolic network, 58
m-Harmonic Green function, 125
Harmonic measure, 102
Harmonic measure at infinity of the section E, 55
Harmonic measure of the point at infinity, 54, 107
m-Harmonic potential, 131
Harmonic support, 57, 65
q-Harmonic support, 95
m-Harmonic support, 128
Harnack property, 47
Harnack property for q-superharmonic functions, 93
H-functions, 88
Hitting time, 2
Homogeneous, 5
H-potential, 88
Hyberbolic network, 4, 49
Hyperplane, 61

Incidence matrix, 1, 21
Infinite graphs, 2
Infinite section, 52
Infinite tree, 16
Inner capacity, 67
Inner flux, 79
Inner normal derivative, 11
Interior, 5
Interior vertex, 46

Kirchhoff's problem, 22

Laplace operator, 2
Laplacian, 7
Lattice, 10
Laurent series, 116
Length, 5, 46
Linear programming problems, 68
Liouville theorem, 19, 71
Lipschitz domains, 60
Locally finite, 5, 46
Locally polar, 129
Logarithmic capacity, 68
Logarithmic kernel, 68, 71
Logarithmic potential, 71, 87

Maria-Frostman maximum principle, 65
Markov chains, 2
Markov property, 2
Martin boundary, 60, 129
Martin compactification, 129
Martin representation, 60, 129
Maximum principle, 47
Maximum principle for finite networks, 25
Minimal, 61, 129
Minimal boundary, 129
Minimal points, 129
Minimum principle, 21, 72
q-Minimum principle, 36
Minimum principle for m-superharmonic functions, 119
Minimum principle for the q-Laplacian, 97
Mixed boundary-value problem, 31
Montel theorem, 120

Neighbours, 5
Network, 5
Neumann problem, 32
Newtonian attraction, 21
Nontangentially accessible domains, 60

Ohm's law, 2
Order, 88
Outer capacity, 68
Outer flux, 78, 79

Outer normal derivative, 11

Parabolic network, 4, 49, 70
Parabolic Riemann surfaces, 3
Path, 5
Perron family, 39, 47
P'-harmonic, 104
Point at infinity, 101
Point biharmonic singularity, 121
Point harmonic support, 9, 113
Poisson equation, 2
Poisson kernel, 31, 62, 129
Poisson problem, 28
q-Poisson solution, 37
Polar set, 67
Polyharmonic function, 109
Polymartingales, 110
Polypotential, 123
Polysuperharmonic, 4
Postman problem, 1
Potential, 48
m-Potential, 123
q-Potential, 94
Potential of f, 63
Potentials with point harmonic support, 56
Pseudo-potentials, 72, 87
m-Potential tree, 124
P-section, 53
P-structure, 103
P'-superharmonic, 104

Random walks, 23
Recurrence, 2
Recurrent, 72
Recurrent tree, 73
Regular exhaustion, 6
Removable singularity, 116
Reversible Markov chain, 2
Riemann surfaces, 3
Riesz representation, 49
Riesz-Martin representation, 130
Riquier problem, 110, 120
Robin problem, 21
Royden compactification, 2
Royden decomposition, 2

Schrödinger operator, 34, 91
Section determined by e and e_i, 52
Self-loop, 5, 46, 111
Simple random walk, 72
Solution to the Kirchhoff's problem, 27
S-section, 53
Standard homogeneous tree, 5
Star domain, 109
Structure subordinate to P, 104
Subharmonic, 7, 46
m-subharmonic, 115
Subharmonic functions of potential-type, 88
Subordinate harmonic structure, 4
Superharmonic, 7, 46
m-superharmonic, 4, 115
q-superharmonic, 37, 91, 92
Symmetric conductance, 5, 46

Terminal, 5, 111
Total inward flux, 12
Transient, 2, 72
Transition probability, 2, 5
Tree, 5

Uniform domains, 60

Vertices, 1
Voltage and current laws, 2

Walk, 5
Wave equations, 2, 45
Weighted graphs, 2

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